Fourier Series

Derivation of Fourier Coefficients

$$\boxed{f(x)} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

} assumes that the period $f(x)$ is $2L$

The corresponding Fourier coefficients are

STEP ONE $\quad a_0 = \frac{1}{L}\int_{2L} f(x)\,dx$

STEP TWO $\quad a_n = \frac{1}{L}\int_{2L} f(x)\cos\frac{n\pi x}{L}\,dx$

STEP THREE $\quad b_n = \frac{1}{L}\int_{2L} f(x)\sin\frac{n\pi x}{L}\,dx$

and integrations are over a single interval in x of $2L$

$f(x)$ has a period 2π

$2L = 2\pi$

$\Rightarrow \boxed{L = \pi}$

$x \rightarrow \frac{\pi x}{L}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right] \qquad eq(1)$$

$$\boxed{\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx \end{aligned}}$$

Useful Integrals:

1) $\int_0^{2\pi} \sin(nx)\,dx = 0$

2) $\int_0^{2\pi} \cos(nx)\,dx = 0$

3) $\int_0^{2\pi} \sin^2(nx)\,dx = \pi$

4) $\int_0^{2\pi} \cos^2(nx)\,dx = \pi$

5) $\int_0^{2\pi} \sin nx \cdot \sin mx\,dx = 0$

6) $\int_0^{2\pi} \cos nx \cdot \cos mx\,dx = 0$

7) $\int_0^{2\pi} \sin nx \cdot \cos mx\,dx = 0$

8) $\int_0^{2\pi} \sin nx \cdot \cos nx\,dx = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \qquad eq(1)$$

1) a_0 : Integrate both sides eq^n (1) from 0 to 2π

$$\int_0^{2\pi} f(x)dx = \frac{a_0}{2}\int_0^{2\pi} dx + a_1 \underbrace{\int_0^{2\pi} \cos x\, dx}_{0} + a_2 \underbrace{\int_0^{2\pi} \cos 2x\, dx}_{0} + \dots + a_n \underbrace{\int_0^{2\pi} \cos nx\, dx}_{0}$$

$$+ b_1 \underbrace{\int_0^{2\pi} \sin x\, dx}_{0} + b_2 \underbrace{\int_0^{2\pi} \sin 2x\, dx}_{0} + \dots + b_n \underbrace{\int_0^{2\pi} \sin nx\, dx}_{0}$$

$$\Rightarrow \int_0^{2\pi} f(x)dx = \frac{a_0}{2}[x]_0^{2\pi}$$

$$\Rightarrow \int_0^{2\pi} f(x)dx = a_0\pi$$

$$\Rightarrow \boxed{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \qquad eq(1)$$

2) a_n : Multiply each of eq(1) by $\cos(nx)$ and integrate from 0 to 2π.

$$\int_0^{2\pi} f(x)\cos(nx)dx = \frac{a_0}{2}\underbrace{\int_0^{2\pi} \cos nx\, dx}_{0} + a_1 \underbrace{\int_0^{2\pi} \cos x \cos nx\, dx}_{0} + \underbrace{\dots}_{0}$$

$$+ a_n \underbrace{\int_0^{2\pi} \cos^2 nx\, dx}_{\pi}$$

$$+ b_1 \underbrace{\int_0^{2\pi} \sin x \cos nx\, dx}_{0} + \dots b_n \underbrace{\int_0^{2\pi} \sin nx \cos nx\, dx}_{0}$$

$$\Rightarrow \int_0^{2\pi} f(x)\cos(nx)dx = a_n\pi$$

$$\Rightarrow \boxed{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \qquad eq(1)$$

3) b_n : Multiply each of eq(1) by $\sin(nx)$ and integrate from 0 to 2π.

$$\int_0^{2\pi} f(x) \sin nx \, dx = \frac{a_0}{2} \underbrace{\int_0^{2\pi} \sin nx \, dx}_{0} + a_1 \underbrace{\int_0^{2\pi} \cos x \cdot \sin nx \, dx}_{0} + \dots +$$

$$a_n \underbrace{\int_0^{2\pi} \cos nx \sin nx \, dx}_{0} +$$

$$b_1 \underbrace{\int_0^{2\pi} \sin x \sin nx \, dx}_{0} + \dots + b_n \underbrace{\int_0^{2\pi} \sin^2 nx \, dx}_{\pi}$$

$$\Rightarrow \int_0^{2\pi} f(x) \sin nx \, dx = b_n \pi$$

$$\Rightarrow \boxed{b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) \, dx}$$