SOLUTIONS

Chapter 9- Canonical Transformation

Book: Classical Mechanics 3rd Edition
Author(s): Herbert Goldstein, Charles P. Poole, John L. Safko

By: Manas Sharma
manassharma07@live.com

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Derivations:

9.4. Show directly that the transformation

$$Q = \log\left(\frac{1}{q}\sin p\right)$$

$$P = q\cot p$$

is canonical.

Sol.9.4.

We are given a transformation as follows,

$$Q = \log\left(\frac{1}{q}\sin p\right)$$

$$P = q\cot p$$

We know that the fundamental **Poisson Brackets** of the transformed variables have the same value when evaluated with respect to any *canonical* coordinate set. In other words ***the fundamental Poisson Brackets are invariant under canonical transformation.***

Therefore, in order for the given transformation to be canonical, the Poisson Bracket of Q, P with respect to q and p should be equal to 1.

Using the formula for **Poisson Bracket**,

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i}$$

$$\therefore [Q, P]_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}$$

$$\implies [Q, P]_{q,p} = \left(\frac{1}{\frac{1}{q}\sin p}\right)\frac{\partial\left(\frac{1}{q}\sin p\right)}{\partial q}(-q\csc^2 p) - \cot p\left(\frac{1}{\frac{1}{q}\sin p}\right)\frac{\partial\left(\frac{1}{q}\sin p\right)}{\partial p}$$

$$\implies [Q, P]_{q,p} = \left(\frac{q}{\sin p}\right)\left(\frac{-\sin p}{q^2}\right)(-q\csc^2 p) - \cot p\left(\frac{q}{\sin p}\right)\left(\frac{1}{q}\cos p\right)$$

$$\implies [Q, P]_{q,p} = \csc^2 p - \cot^2 p$$

$$\implies [Q, P]_{q,p} = 1$$

$$\therefore \boxed{[Q, P]_{q,p} = [Q, P]_{Q,P}}$$

Hence Proved.

9.5. Show directly that for a system of one degree of freedom, the transformation

$$Q = \arctan\frac{\alpha q}{p}$$

$$P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)$$

is canonical, where α is an arbitrary constant of suitable dimensions.

Sol.9.5. We are given a transformation as follows,

$$Q = \arctan\frac{\alpha q}{p}$$

$$P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)$$

We know that the fundamental **Poisson Brackets** of the transformed variables have the same value when evaluated with respect to any *canonical* coordinate set. In other words ***the fundamental Poisson Brackets are invariant under canonical transformation.***
Therefore, in order, for the given transformation to be canonical, the Poisson Bracket of Q, P with respect to q and p should be equal to 1.
Using the formula for **Poisson Bracket**,

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i}$$

$$\therefore [Q, P]_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}$$

$$\implies [Q, P]_{q,p} = \left(\frac{1}{1+\frac{\alpha^2 q^2}{p^2}}\right)\frac{\partial\left(\frac{\alpha q}{p}\right)}{\partial q}\left(\frac{p}{\alpha}\right) - 2\left(\frac{\alpha q}{2}\right)\left(\frac{1}{1+\frac{\alpha^2 q^2}{p^2}}\right)\frac{\partial\left(\frac{\alpha q}{p}\right)}{\partial p}$$

$$\implies [Q, P]_{q,p} = \left(\frac{1}{1+\frac{\alpha^2 q^2}{p^2}}\right)\left(\frac{\alpha}{p}\right)\left(\frac{p}{\alpha}\right) - (\alpha q)\left(\frac{1}{1+\frac{\alpha^2 q^2}{p^2}}\right)\left(\frac{-\alpha q}{p^2}\right)$$

$$\implies [Q, P]_{q,p} = \left(\frac{p^2}{p^2+\alpha^2 q^2}\right)\left[1+\frac{\alpha^2 q^2}{p^2}\right]$$

$$\implies [Q, P]_{q,p} = \left(\frac{p^2}{p^2+\alpha^2 q^2}\right)\left(\frac{p^2+\alpha^2 q^2}{p^2}\right)$$

$$\implies [Q, P]_{q,p} = 1$$

$$\therefore \boxed{[Q, P]_{q,p} = [Q, P]_{Q,P}}$$

Hence Proved.

9.6 The transformation equations between two sets of coordinates are

$$Q = \log(1 + \sqrt{q}\cos p)$$

$$P = 2(1 + \sqrt{q}\cos p)\sqrt{q}\sin p$$

(a) Show directly from these transformation equations that Q, P are canonical variables if q and p are.
(b) Show that the function that generates this transformation is

$$F_3 = -(e^Q - 1)^2 \tan p$$

Sol.9.6. (a)
We are given a transformation as follows,

$$Q = \log(1 + \sqrt{q}\cos p)$$

$$P = 2(1 + \sqrt{q}\cos p)\sqrt{q}\sin p$$

We can re-write the second equation as:

$$\implies P = 2\sqrt{q}\sin p + 2q\cos p\sin p$$

We know that the fundamental **Poisson Brackets** of the transformed variables have the same value when evaluated with respect to any *canonical* coordinate set. In other words ***the fundamental Poisson Brackets are invariant under canonical transformation.***
Therefore, in order, for the given transformation to be canonical, the Poisson Bracket of Q, P with respect to q and p should be equal to 1.

Using the formula for **Poisson Bracket**,

$$[u,v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i}$$

$$\therefore [Q,P]_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}$$
$$\implies [Q,P]_{q,p} = \left(\frac{1}{1+\sqrt{q}cosp}\right)\frac{\partial\left(1+\sqrt{q}cosp\right)}{\partial q}\left[2\sqrt{q}cosp + 2q\frac{\partial(cospsinp)}{\partial p}\right] - \left[2sinp\left(\frac{1}{2\sqrt{q}}\right)+2cospsinp\right]\left(\frac{1}{1+\sqrt{q}cosp}\right)\frac{\partial\left(1+\sqrt{q}cosp\right)}{\partial p}$$
$$\implies [Q,P]_{q,p} = \left(\frac{1}{1+\sqrt{q}cosp}\right)\left(\frac{cosp}{2\sqrt{q}}\right)\left[2\sqrt{q}cosp + 2qcos^2p - 2qsin^2p\right] - \left[2sinp\left(\frac{1}{2\sqrt{q}}\right)+2cospsinp\right]\left(\frac{1}{1+\sqrt{q}cosp}\right)(-\sqrt{q}sinp)$$
$$\implies [Q,P]_{q,p} = \left(\frac{1}{1+\sqrt{q}cosp}\right)\left[cos^2p + \sqrt{q}cos^3p - \sqrt{q}cospsin^2p\right] + \left(\frac{1}{1+\sqrt{q}cosp}\right)\left[sin^2p + 2\sqrt{q}cospsin^2p\right]$$
$$\implies [Q,P]_{q,p} = \left(\frac{1}{1+\sqrt{q}cosp}\right)\left[cos^2p + sin^2p + \sqrt{q}cos^3p - \sqrt{q}cospsin^2p + 2\sqrt{q}cospsin^2p\right]$$
$$\implies [Q,P]_{q,p} = \left(\frac{1}{1+\sqrt{q}cosp}\right)\left[1 + \sqrt{q}cosp\left(cos^2p - sin^2p + 2sin^2p\right)\right]$$
$$\implies [Q,P]_{q,p} = \left(\frac{1}{1+\sqrt{q}cosp}\right)\left[1 + \sqrt{q}cosp\left(cos^2p + sin^2p\right)\right]$$
$$\implies [Q,P]_{q,p} = \left(\frac{1}{1+\sqrt{q}cosp}\right)(1+\sqrt{q}cosp)$$
$$\implies [Q,P]_{q,p} = 1$$
$$\therefore \boxed{[Q,P]_{q,p} = [Q,P]_{Q,P}}$$

Hence Proved.

(b) We are given the following generating function of the F_3 type:

$$F_3 = -(e^Q - 1)^2 \tan p$$

For a generating function of F_3 type, q and p are given as:

$$q = -\frac{\partial F_3}{\partial p}$$
$$\implies q = (e^Q-1)^2 \sec^2 p$$
$$\implies \sqrt{q}\cos p = e^Q - 1$$
$$\implies \boxed{Q = \log(1+\sqrt{q}\cos p)}$$

$$P = -\frac{\partial F_3}{\partial Q}$$
$$\implies P = 2\tan p(e^Q-1)e^Q$$
$$\implies P = 2\tan p(1+\sqrt{q}\cos p - 1)(1+\sqrt{q}\cos p)$$
$$\implies P = 2\frac{\sin p}{\cos p}\sqrt{q}\cos p(1+\sqrt{q}\cos p)$$
$$\implies \boxed{P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p}$$

Therefore, the given generating function does in fact generate the given transformation.

9.8. Prove directly that the transformation

$$Q_1 = q_1 \qquad P_1 = p_1 - 2p_2$$
$$Q_2 = p_2 \qquad P_2 = -2q_1 - q_2$$

is canonical and find a generating function.
Sol.9.8. We are given a transformation as follows,

$$Q_1 = q_1 \qquad P_1 = p_1 - 2p_2$$
$$Q_2 = p_2 \qquad P_2 = -2q_1 - q_2$$

We know that the fundamental **Poisson Brackets** of the transformed variables have the same value when evaluated with respect to any *canonical* coordinate set. In other words ***the fundamental Poisson Brackets are invariant under canonical transformation.***
Therefore, in order, for the given transformation, to be canonical, the Poisson Bracket of Q, P with respect to q and p should satisfy the following:

$$[Q_j, P_k]_{q,p} = \delta_{jk}$$

i.e. we need to prove,

$$[Q_1, P_1]_{q,p} = 1,$$
$$[Q_2, P_2]_{q,p} = 1$$

and,

$$[Q_1, Q_2]_{q,p} = [P_1, P_2]_{q,p} = [Q_1, P_2]_{q,p} = [Q_2, P_1]_{q,p} = 0$$

Using the formula for **Poisson Bracket**,

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i}$$

$$\therefore [Q_1, P_1]_{q,p} = \frac{\partial Q_1}{\partial q_i}\frac{\partial P_1}{\partial p_i} - \frac{\partial P_1}{\partial q_i}\frac{\partial Q_1}{\partial p_i}$$
$$\implies [Q_1, P_1]_{q,p} = \frac{\partial Q_1}{\partial q_1}\frac{\partial P_1}{\partial p_1} - \frac{\partial P_1}{\partial q_1}\frac{\partial Q_1}{\partial p_1} + \frac{\partial Q_1}{\partial q_2}\frac{\partial P_1}{\partial p_2} - \frac{\partial P_1}{\partial q_2}\frac{\partial Q_1}{\partial p_2}$$
$$\implies [Q_1, P_1]_{q,p} = 1 - 0 + 0 - 0$$
$$\implies \boxed{[Q_1, P_1]_{q,p} = 1}$$

Now,

$$[Q_2, P_2]_{q,p} = \frac{\partial Q_2}{\partial q_i}\frac{\partial P_2}{\partial p_i} - \frac{\partial P_2}{\partial q_i}\frac{\partial Q_2}{\partial p_i}$$
$$\implies [Q_2, P_2]_{q,p} = \frac{\partial Q_2}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial P_2}{\partial q_1}\frac{\partial Q_2}{\partial p_1} + \frac{\partial Q_2}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial P_2}{\partial q_2}\frac{\partial Q_2}{\partial p_2}$$
$$\implies [Q_2, P_2]_{q,p} = 0 - 0 + 0 - (-1)1$$
$$\implies \boxed{[Q_2, P_2]_{q,p} = 1}$$

$$[Q_1, Q_2]_{q,p} = \frac{\partial Q_1}{\partial q_i}\frac{\partial Q_2}{\partial p_i} - \frac{\partial Q_2}{\partial q_i}\frac{\partial Q_1}{\partial p_i}$$
$$\implies [Q_1, Q_2]_{q,p} = \frac{\partial Q_1}{\partial q_1}\frac{\partial Q_2}{\partial p_1} - \frac{\partial Q_2}{\partial q_1}\frac{\partial Q_1}{\partial p_1} + \frac{\partial Q_1}{\partial q_2}\frac{\partial Q_2}{\partial p_2} - \frac{\partial Q_2}{\partial q_2}\frac{\partial Q_1}{\partial p_2}$$
$$\implies [Q_1, Q_2]_{q,p} = 0 - 0 + 0 - 0$$
$$\implies \boxed{[Q_1, Q_2]_{q,p} = 0}$$

$$
\begin{aligned}
&[P_1, P_2]_{q,p} = \frac{\partial P_1}{\partial q_i}\frac{\partial P_2}{\partial p_i} - \frac{\partial P_2}{\partial q_i}\frac{\partial P_1}{\partial p_i} \\
\implies &[P_1, P_2]_{q,p} = \frac{\partial P_1}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial P_2}{\partial q_1}\frac{\partial P_1}{\partial p_1} + \frac{\partial P_1}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial P_2}{\partial q_2}\frac{\partial P_1}{\partial p_2} \\
\implies &[P_1, P_2]_{q,p} = 0 - (-2) + 0 - (-1)(-2) \\
\implies &\boxed{[P_1, P_2]_{q,p} = 0}
\end{aligned}
$$

$$
\begin{aligned}
&[Q_1, P_2]_{q,p} = \frac{\partial Q_1}{\partial q_i}\frac{\partial P_2}{\partial p_i} - \frac{\partial P_2}{\partial q_i}\frac{\partial Q_1}{\partial p_i} \\
\implies &[Q_1, P_2]_{q,p} = \frac{\partial Q_1}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial P_2}{\partial q_1}\frac{\partial Q_1}{\partial p_1} + \frac{\partial Q_1}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial P_2}{\partial q_2}\frac{\partial Q_1}{\partial p_2} \\
\implies &[Q_1, P_2]_{q,p} = 0 - 0 + 0 - 0 \\
\implies &\boxed{[Q_1, P_2]_{q,p} = 0}
\end{aligned}
$$

$$
\begin{aligned}
&[Q_2, P_1]_{q,p} = \frac{\partial Q_2}{\partial q_i}\frac{\partial P_1}{\partial p_i} - \frac{\partial P_1}{\partial q_i}\frac{\partial Q_2}{\partial p_i} \\
\implies &[Q_2, P_1]_{q,p} = \frac{\partial Q_2}{\partial q_1}\frac{\partial P_1}{\partial p_1} - \frac{\partial P_1}{\partial q_1}\frac{\partial Q_2}{\partial p_1} + \frac{\partial Q_2}{\partial q_2}\frac{\partial P_1}{\partial p_2} - \frac{\partial P_1}{\partial q_2}\frac{\partial Q_2}{\partial p_2} \\
\implies &[Q_2, P_1]_{q,p} = 0 - 0 + 0 - 0 \\
\implies &\boxed{[Q_2, P_1]_{q,p} = 0}
\end{aligned}
$$

Therefore the given transformation is canonical.

Now, we are also required to find a generating function for the given transformation.

For that, the first thing we need to do is to find the form of the generating function.
Finding a suitable form of the generating function would require some analysis along with a lot of hit-and-trial.

What I usually do is, I assume the generating function to be of the first kind, i.e. $F = F_1(q, Q, t)$ and then find the form of the generating function by integrating it's corresponding partial derivatives. Now if the generating function is, in fact, of the first kind then you would be able to find it in the form of q and Q(may require a few substitutions using the given transformation equations).
But if it were of some other form then, no matter what, you won't be able to write in the form of q and Q.
So now we look for the coordinates that can't be expressed in the form of q and Q and will have to be included in the generating function.
Once you have such coordinates, then we have a form of our generating function.

Note: The form of your generating function should consist of equal no. of old and new coordinates.
Now after a little hit-and-trial, I found the generating function to be of the mixed form, $F'(p_1, q_2, Q_1, Q_2)$.
Now the equations of transformation can be found by writing F as

$$F = F'(p_1, q_2, Q_1, Q_2) + q_1 p_1$$

and plugging it in the following equation(from Hamilton's Principle):

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt}$$

Expanding the derivative of F

,

$$\implies p_1\dot{q_1} + p_2\dot{q_2} - H = P_1\dot{Q}_1 + P_2\dot{Q}_2 - K + \frac{\partial F'}{\partial t} + \frac{\partial F}{\partial p_1}\dot{p_1} + \frac{\partial F}{\partial q_2}\dot{q_2} + \frac{\partial F}{\partial Q_1}\dot{Q}_1 + \frac{\partial F}{\partial Q_2}\dot{Q}_2 + p_1\dot{q_1} + \dot{p_1}q_1$$

and equating the coefficients of $\dot{p_1}$, $\dot{q_2}$, $\dot{Q}_1$ and $\dot{Q}_2$ to zero, leads to the equations

$$P_1 = -\frac{\partial F'}{\partial Q_1} \qquad p_2 = \frac{\partial F'}{\partial q_2}$$

$$P_2 = -\frac{\partial F'}{\partial Q_2} \qquad q_1 = -\frac{\partial F'}{\partial p_1}$$

integrating all the above equations we get,

$$\begin{aligned}
&P_1 = -\frac{\partial F'}{\partial Q_1}\\
\implies &F' = -P_1Q_1 + c_1\\
\implies &F' = -(p_1 - 2p_2)Q_1 + c_1\\
\implies &F' = -(p_1 - 2Q_2)Q_1 + c_1\\
\implies &\boxed{F' = -p_1Q_1 + 2Q_1Q_2 + c_1}\\
&P_2 = -\frac{\partial F'}{\partial Q_2}\\
\implies &F' = -P_2Q_2 + c_2\\
\implies &F' = -(-2q_1 - q_2)Q_2 + c_2\\
\implies &F' = -(-2Q_1 - q_2)Q_2 + c_2\\
\implies &\boxed{F' = 2Q_1Q_2 + q_2Q_2 + c_2}\\
&p_2 = \frac{\partial F'}{\partial q_2}\\
\implies &F' = p_q q_2 + c_3\\
\implies &\boxed{F' = q_2Q_2 + c_3}\\
&q_1 = -\frac{\partial F'}{\partial p_1}\\
\implies &F' = -q_1p_1 + c_4\\
\implies &\boxed{F' = -p_1Q_1 + 2Q_1Q_2 + c_4}
\end{aligned}$$

Clearly, the terms $-p_1Q_1$, $2Q_1Q_2$ and q_2Q_2 repeat themselves more than once, so they are probably present in the generating function, F'.
If we use $c_1 = q_2Q_2$, $c_2 = -p_1Q_1$, $c_3 = 2Q_1Q_2 - p_1Q_1$, and $c_4 = q_2Q_2$ then F' gives back the canonical transformation, Therefore,

$$\boxed{F' = 2Q_1Q_2 + q_2Q_2 - p_1Q_1}$$

is the generating function for the given canonical transformation.

You should verify if you get the transformation using this generating function.

9.10. Find under what conditions

$$Q = \frac{\alpha p}{x} \qquad\qquad P = \beta x^2$$

where α and β are constants, represents a canonical transformation for a system of one degree of freedom, and obtain a suitable generating function. Apply the transformation to the solution of the linear harmonic oscillator.

Sol.9.10

We are given a transformation as follows,

$$Q = \frac{\alpha p}{x}$$
$$P = \beta x^2$$

We know that the fundamental **Poisson Brackets** of the transformed variables have the same value when evaluated with respect to any *canonical* coordinate set. In other words ***the fundamental Poisson Brackets are invariant under canonical transformation.***

Therefore, in order, for the given transformation to be canonical, the Poisson Bracket of Q, P with respect to q and p should be equal to 1.

Using the formula for **Poisson Bracket**,

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i}$$

$$\therefore [Q, P]_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}$$
$$\implies [Q, P]_{q,p} = \left(\frac{-\alpha p}{x^2}\right)0 - 2\beta x\frac{\alpha}{x}$$
$$\implies [Q, P]_{q,p} = -2\beta\alpha$$

But for the transformation to be canonical:

$$[Q, P]_{q,p} = 1$$

$$\therefore -2\beta\alpha = 1$$
$$\implies \boxed{\beta = -\frac{1}{2\alpha}}$$

9.11. Determine whether the transformation

$$Q_1 = q_1 q_2 \qquad\qquad P_1 = \frac{p_1 - p_2}{q_2 - q_1} + 1$$
$$Q_2 = q_1 + q_2 \qquad\qquad P_2 = -\frac{q_2 p_2 - q_1 p_1}{q_2 - q_1} - (q_2 + q_1)$$

is canonical.

Sol.9.11.

We are given a transformation as follows,

$$Q_1 = q_1 q_2 \qquad\qquad P_1 = \frac{p_1 - p_2}{q_2 - q_1} + 1$$
$$Q_2 = q_1 + q_2 \qquad\qquad P_2 = -\frac{q_2 p_2 - q_1 p_1}{q_2 - q_1} - (q_2 + q_1)$$

We know that the fundamental **Poisson Brackets** of the transformed variables have the same value when evaluated with respect to any *canonical* coordinate set. In other words ***the fundamental Poisson Brackets are invariant under canonical transformation.***
Therefore, in order, for the given transformation, to be canonical, the Poisson Bracket of Q, P with respect to q and p should satisfy the following:

$$[Q_j, P_k]_{q,p} = \delta_{jk}$$

i.e. we need to prove,

$$[Q_1, P_1]_{q,p} = 1$$

,

$$[Q_2, P_2]_{q,p} = 1$$

and,

$$[Q_1, Q_2]_{q,p} = [P_1, P_2]_{q,p} = [Q_1, P_2]_{q,p} = [Q_2, P_1]_{q,p} = 0$$

Using the formula for **Poisson Bracket**,

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i}$$

$$\begin{aligned}
&[Q_1, P_1]_{q,p} = \frac{\partial Q_1}{\partial q_i}\frac{\partial P_1}{\partial p_i} - \frac{\partial P_1}{\partial q_i}\frac{\partial Q_1}{\partial p_i} \\
\implies &[Q_1, P_1]_{q,p} = \frac{\partial Q_1}{\partial q_1}\frac{\partial P_1}{\partial p_1} - \frac{\partial P_1}{\partial q_1}\frac{\partial Q_1}{\partial p_1} + \frac{\partial Q_1}{\partial q_2}\frac{\partial P_1}{\partial p_2} - \frac{\partial P_1}{\partial q_2}\frac{\partial Q_1}{\partial p_2} \\
\implies &[Q_1, P_1]_{q,p} = q_2\left(\frac{1}{q_2 - q_1}\right) - 0 + q_1\left(\frac{-1}{q_2 - q_1}\right) \\
\implies &[Q_1, P_1]_{q,p} = \frac{q_2 - q_1}{q_2 - q_1} \\
\implies &\boxed{[Q_1, P_1]_{q,p} = 1}
\end{aligned}$$

Now,

$$\begin{aligned}
&[Q_2, P_2]_{q,p} = \frac{\partial Q_2}{\partial q_i}\frac{\partial P_2}{\partial p_i} - \frac{\partial P_2}{\partial q_i}\frac{\partial Q_2}{\partial p_i} \\
\implies &[Q_2, P_2]_{q,p} = \frac{\partial Q_2}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial P_2}{\partial q_1}\frac{\partial Q_2}{\partial p_1} + \frac{\partial Q_2}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial P_2}{\partial q_2}\frac{\partial Q_2}{\partial p_2} \\
\implies &[Q_2, P_2]_{q,p} = \frac{-q_1}{q_2 - q_1} - 0 + \frac{q_2}{q_2 - q_1} - 0 \\
\implies &[Q_2, P_2]_{q,p} = \frac{q_2 - q_1}{q_2 - q_1} \\
\implies &\boxed{[Q_2, P_2]_{q,p} = 1}
\end{aligned}$$

$$\begin{aligned}
&[Q_1, Q_2]_{q,p} = \frac{\partial Q_1}{\partial q_i}\frac{\partial Q_2}{\partial p_i} - \frac{\partial Q_2}{\partial q_i}\frac{\partial Q_1}{\partial p_i} \\
\implies &[Q_1, Q_2]_{q,p} = \frac{\partial Q_1}{\partial q_1}\frac{\partial Q_2}{\partial p_1} - \frac{\partial Q_2}{\partial q_1}\frac{\partial Q_1}{\partial p_1} + \frac{\partial Q_1}{\partial q_2}\frac{\partial Q_2}{\partial p_2} - \frac{\partial Q_2}{\partial q_2}\frac{\partial Q_1}{\partial p_2} \\
\implies &[Q_1, Q_2]_{q,p} = 0 - 0 + 0 - 0 \\
\implies &\boxed{[Q_1, Q_2]_{q,p} = 0}
\end{aligned}$$

$$
\begin{aligned}
&[P_1, P_2]_{q,p} = \frac{\partial P_1}{\partial q_i}\frac{\partial P_2}{\partial p_i} - \frac{\partial P_2}{\partial q_i}\frac{\partial P_1}{\partial p_i} \\
\implies &[P_1, P_2]_{q,p} = \frac{\partial P_1}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial P_2}{\partial q_1}\frac{\partial P_1}{\partial p_1} + \frac{\partial P_1}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial P_2}{\partial q_2}\frac{\partial P_1}{\partial p_2} \\
\implies &[P_1, P_2]_{q,p} = \left(\frac{p_1 - p_2}{(q_2 - q_1)^2}\right)\left(\frac{q_1}{q_2 - q_1}\right) \left[\frac{(q_1p_1 - q_2p_2)}{(q_2 - q_1)^2} + \frac{p_1}{q_2 - q_1} - 1\right]\left(\frac{1}{q_2 - q_1}\right) \\
&\qquad + \left(-\frac{p_1 - p_2}{(q_2 - q_1)^2}\right)\left(\frac{-q_2}{q_2 - q_1}\right) - \left[\frac{(q_2p_2 - q_1p_1)}{(q_2 - q_1)^2} - \frac{p_2}{q_2 - q_1} - 1\right]\left(-\frac{1}{q_2 - q_1}\right) \\
\implies &[P_1, P_2]_{q,p} = \frac{p_1q_1 - p_2q_1}{(q_2 - q_1)^3} - \frac{p_1q_1 - p_2q_2}{(q_2 - q_1)^3} - \frac{p_1}{(q_2 - q_1)^2} + \frac{1}{q_2 - q_1} \\
&\qquad + \frac{p_1q_2 - p_2q_2}{(q_2 - q_1)^3} - \frac{p_2q_2 - p_1q_1}{(q_2 - q_1)^3} - \frac{p_2}{(q_2 - q_1)^2} - \frac{1}{q_2 - q_1} \\
\implies &[P_1, P_2]_{q,p} = \frac{p_1q_1 - p_2q_1 - p_1q_1 + p_2q_2 - p_1q_2 + p_1q_1 + p_1q_2 - p_2q_2 + q_2p_2 - p_1q_1 - p_2q_2 + p_2q_1}{(q_2 - q_1)^3} \\
\implies &\boxed{[P_1, P_2]_{q,p} = 0}
\end{aligned}
$$

$$
\begin{aligned}
&[Q_1, P_2]_{q,p} = \frac{\partial Q_1}{\partial q_i}\frac{\partial P_2}{\partial p_i} - \frac{\partial P_2}{\partial q_i}\frac{\partial Q_1}{\partial p_i} \\
\implies &[Q_1, P_2]_{q,p} = \frac{\partial Q_1}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial P_2}{\partial q_1}\frac{\partial Q_1}{\partial p_1} + \frac{\partial Q_1}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial P_2}{\partial q_2}\frac{\partial Q_1}{\partial p_2} \\
\implies &[Q_1, P_2]_{q,p} = q_2\left(\frac{q_1}{q_2 - q_1}\right) - 0 + q_1\left(\frac{-q_2}{q_2 - q_1}\right) - 0 \\
\implies &[Q_1, P_2]_{q,p} = \frac{q_2q_1 - q_1q_2}{q_2 - q_1} \\
\implies &\boxed{[Q_1, P_2]_{q,p} = 0}
\end{aligned}
$$

$$
\begin{aligned}
&[Q_2, P_1]_{q,p} = \frac{\partial Q_2}{\partial q_i}\frac{\partial P_1}{\partial p_i} - \frac{\partial P_1}{\partial q_i}\frac{\partial Q_2}{\partial p_i} \\
\implies &[Q_2, P_1]_{q,p} = \frac{\partial Q_2}{\partial q_1}\frac{\partial P_1}{\partial p_1} - \frac{\partial P_1}{\partial q_1}\frac{\partial Q_2}{\partial p_1} + \frac{\partial Q_2}{\partial q_2}\frac{\partial P_1}{\partial p_2} - \frac{\partial P_1}{\partial q_2}\frac{\partial Q_2}{\partial p_2} \\
\implies &[Q_2, P_1]_{q,p} = \frac{1}{q_2 - q_1} - 0 + \frac{-1}{q_2 - q_1} - 0 \\
\implies &\boxed{[Q_2, P_1]_{q,p} = 0}
\end{aligned}
$$

Therefore the given transformation is canonical.

9.14. Prove that the transformation

$$Q_1 = q_1^2 \qquad P_1 = \frac{p_1 \cos p_2 - 2q_2}{2q_1 \cos p_2}$$

$$Q_2 = q_2 \sec p_2 \qquad P_2 = \sin p_2 - 2q_1$$

is canonical, by any method you choose. Find a suitable generating function that will lead to this transformation.

Sol.9.14.

We are given a transformation as follows,

$$Q_1 = q_1^2 \qquad P_1 = \frac{p_1 \cos p_2 - 2q_2}{2q_1 \cos p_2}$$

$$Q_2 = q_2 \sec p_2 \qquad P_2 = \sin p_2 - 2q_1$$

We know that the fundamental **Poisson Brackets** of the transformed variables have the same value when evaluated with respect to any *canonical* coordinate set. In other words ***the fundamental Poisson Brackets are invariant under canonical transformation.***
Therefore, in order, for the given transformation, to be canonical, the Poisson Bracket of Q, P with respect to q and p should satisfy the following:

$$[Q_j, P_k]_{q,p} = \delta_{jk}$$

i.e. we need to prove,

$$[Q_1, P_1]_{q,p} = 1$$

,

$$[Q_2, P_2]_{q,p} = 1$$

and,

$$[Q_1, Q_2]_{q,p} = [P_1, P_2]_{q,p} = [Q_1, P_2]_{q,p} = [Q_2, P_1]_{q,p} = 0$$

Using the formula for **Poisson Bracket**,

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i}$$

$$\begin{aligned}
[Q_1, P_1]_{q,p} &= \frac{\partial Q_1}{\partial q_i}\frac{\partial P_1}{\partial p_i} - \frac{\partial P_1}{\partial q_i}\frac{\partial Q_1}{\partial p_i} \\
\implies [Q_1, P_1]_{q,p} &= \frac{\partial Q_1}{\partial q_1}\frac{\partial P_1}{\partial p_1} - \frac{\partial P_1}{\partial q_1}\frac{\partial Q_1}{\partial p_1} + \frac{\partial Q_1}{\partial q_2}\frac{\partial P_1}{\partial p_2} - \frac{\partial P_1}{\partial q_2}\frac{\partial Q_1}{\partial p_2} \\
\implies [Q_1, P_1]_{q,p} &= 2q_1\frac{\cos p_2}{2q_1\cos p_2} - 0 + 0 - 0 \\
\implies & \boxed{[Q_1, P_1]_{q,p} = 1}
\end{aligned}$$

$$\begin{aligned}
[Q_2, P_2]_{q,p} &= \frac{\partial Q_2}{\partial q_i}\frac{\partial P_2}{\partial p_i} - \frac{\partial P_2}{\partial q_i}\frac{\partial Q_2}{\partial p_i} \\
\implies [Q_2, P_2]_{q,p} &= \frac{\partial Q_2}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial P_2}{\partial q_1}\frac{\partial Q_2}{\partial p_1} + \frac{\partial Q_2}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial P_2}{\partial q_2}\frac{\partial Q_2}{\partial p_2} \\
\implies [Q_2, P_2]_{q,p} &= 0 - 0 + \sec p_2 \cos p_2 - 0 \\
\implies & \boxed{[Q_2, P_2]_{q,p} = 1}
\end{aligned}$$

$$\begin{aligned}
[Q_1, Q_2]_{q,p} &= \frac{\partial Q_1}{\partial q_i}\frac{\partial Q_2}{\partial p_i} - \frac{\partial Q_2}{\partial q_i}\frac{\partial Q_1}{\partial p_i} \\
\implies [Q_1, Q_2]_{q,p} &= \frac{\partial Q_1}{\partial q_1}\frac{\partial Q_2}{\partial p_1} - \frac{\partial Q_2}{\partial q_1}\frac{\partial Q_1}{\partial p_1} + \frac{\partial Q_1}{\partial q_2}\frac{\partial Q_2}{\partial p_2} - \frac{\partial Q_2}{\partial q_2}\frac{\partial Q_1}{\partial p_2} \\
\implies [Q_1, Q_2]_{q,p} &= 0 - 0 + 0 - 0 \\
\implies & \boxed{[Q_1, Q_2]_{q,p} = 0}
\end{aligned}$$

$$\begin{aligned}
[P_1, P_2]_{q,p} &= \frac{\partial P_1}{\partial q_i}\frac{\partial P_2}{\partial p_i} - \frac{\partial P_2}{\partial q_i}\frac{\partial P_1}{\partial p_i} \\
\implies [P_1, P_2]_{q,p} &= \frac{\partial P_1}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial P_2}{\partial q_1}\frac{\partial P_1}{\partial p_1} + \frac{\partial P_1}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial P_2}{\partial q_2}\frac{\partial P_1}{\partial p_2} \\
\implies [P_1, P_2]_{q,p} &= 0 - \left(\frac{\cos p_2}{2q_1\cos p_2}\right)(-2) + \left(\frac{-2}{2q_1\cos p_2}\right)(\cos p_2) - 0 \\
\implies & \boxed{[P_1, P_2]_{q,p} = 0}
\end{aligned}$$

$$
\begin{aligned}
[Q_1,P_2]_{q,p} &= \frac{\partial Q_1}{\partial q_i}\frac{\partial P_2}{\partial p_i} - \frac{\partial P_2}{\partial q_i}\frac{\partial Q_1}{\partial p_i} \\
\implies [Q_1,P_2]_{q,p} &= \frac{\partial Q_1}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial P_2}{\partial q_1}\frac{\partial Q_1}{\partial p_1} + \frac{\partial Q_1}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial P_2}{\partial q_2}\frac{\partial Q_1}{\partial p_2} \\
\implies [Q_1,P_2]_{q,p} &= 0 - 0 + 0 - 0 \\
\implies & \boxed{[Q_1,P_2]_{q,p} = 0}
\end{aligned}
$$

$$
\begin{aligned}
[Q_2,P_1]_{q,p} &= \frac{\partial Q_2}{\partial q_i}\frac{\partial P_1}{\partial p_i} - \frac{\partial P_1}{\partial q_i}\frac{\partial Q_2}{\partial p_i} \\
\implies [Q_2,P_1]_{q,p} &= \frac{\partial Q_2}{\partial q_1}\frac{\partial P_1}{\partial p_1} - \frac{\partial P_1}{\partial q_1}\frac{\partial Q_2}{\partial p_1} + \frac{\partial Q_2}{\partial q_2}\frac{\partial P_1}{\partial p_2} - \frac{\partial P_1}{\partial q_2}\frac{\partial Q_2}{\partial p_2} \\
\implies [Q_2,P_1]_{q,p} &= 0 - 0 + \sec p_2\left(\frac{-q_2}{q_1}\sec p_2 \tan p_2\right) - \left(\frac{-\sec p_2}{q_1}\right) q_2 \sec p_2 \tan p_2 \\
\implies & \boxed{[Q_2,P_1]_{q,p} = 0}
\end{aligned}
$$

Therefore the given transformation is canonical.

Now, we are also required to find a generating function for the given transformation.
After a little hit-and-trial, I found the generating function to be of the fourth kind,

$$F_4(p_1, p_2, Q_1, Q_2)$$

Now the equations of transformation can be found by writing F as

$$F = F_4(p_1, q_2, Q_1, Q_2) + p_1 q_1 + p_2 q_2$$

and plugging it in the following equation(from Hamilton's Principle):

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt}$$

Expanding the derivative of F_4 and equating the coefficients of $\dot{q}_i$ and $\dot{P}_i$ to zero, we get the equations

$$
P_1 = -\frac{\partial F_4}{\partial Q_1} \qquad q_1 = -\frac{\partial F_4}{\partial p_1}
$$

$$
P_2 = -\frac{\partial F_4}{\partial Q_2} \qquad q_2 = -\frac{\partial F_4}{\partial p_2}
$$

integrating all the above equations we get,

$$q_1 = -\frac{\partial F_4}{\partial p_1}$$
$$\implies F_4 = -q_1 p_1 + c_1$$
$$\implies \boxed{F_4 = -\sqrt{Q_1} p_1 + c_1}$$
$$q_2 = -\frac{\partial F_4}{\partial p_2}$$
$$\implies F_4 = -q_2 p_2 + c_2$$
$$\implies \boxed{F_4 = -Q_2 \cos p_2 p_2 + c_2}$$
$$P_1 = -\frac{\partial F_4}{\partial Q_1}$$
$$\implies F_4 = -P_1 Q_1 + c_3$$
$$\implies F_4 = -\left(\frac{p_1 \cos p_2 - 2q_2}{2q_1 \cos p_2}\right) Q_1 + c_3$$
$$\implies \boxed{F_4 = -\frac{p_1\sqrt{Q_1}}{2} + Q_2\sqrt{Q_1} + c_3}$$
$$P_2 = -\frac{\partial F_4}{\partial Q_2}$$
$$\implies F_4 = -P_2 Q_2 + c_4$$
$$\implies \boxed{F_4 = -Q_2 \sin p_2 + 2\sqrt{Q_1} Q_2 + c_4}$$

Using,

$$c_4 = -\sqrt{Q_1} p_1$$

we get back the canonical transformation.
Therefore,

$$\boxed{F_4 = 2\sqrt{Q_1} Q_2 - \sqrt{Q_1} p_1 - Q_2 \sin p_2}$$

is the required generating function.
You can find the remaining constants if you want to but there is no need as we have already found the generating function. You should verify if you get canonical transformation using the above generating function.

9.17. Show that the Jacobi identity is satisfied if the Poisson Bracket sign stands for the commutator of two square matrices.

$$[A, B] = AB - BA$$

Show also that for the same representation of the Poisson bracket that

$$[A, BC] = [A, B]C + B[A, C]$$

Sol.9.17.
We are given that the Poisson bracket sign stands for the commutator of two square matrices:

$$[A, B] = AB - BA$$

We need to show that the Jacobi identity is satisfied by the new(above) definition of Poisson Brackets.
Therefore, we need to prove:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

Proof:

From the new definition of P.B.:

$$\begin{aligned}
&[A,[B,C]] = [A, BC - CB] \\
\implies &[A,[B,C]] = A(BC - CB) - (BC - CB)A \\
\implies &[A,[B,C]] = ABC - ACB - BCA + CBA \qquad (1)
\end{aligned}$$

Note: We cant simply cancel ABC by ACB as these are matrices and hence their product isnt commutative.
Similarly,

$$\begin{aligned}
&[B,[C,A]] = [B, CA - AC] \\
\implies &[B,[C,A]] = B(CA - AC) - (CA - AC)B \\
\implies &[B,[C,A]] = BCA - BAC - CAB + ACB \qquad (2)
\end{aligned}$$

and,

$$\begin{aligned}
&[C,[A,B]] = [C, AB - BA] \\
\implies &[C,[A,B]] = C(AB - BA) - (AB - BA)C \\
\implies &[C,[A,B]] = CAB - CBA - ABC + BAC \qquad (3)
\end{aligned}$$

Adding (1), (2), and (3), we get

$$\begin{aligned}
[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = ABC - ACB - BCA + CBA + BCA \\
- BAC - CAB + ACB + CAB - CBA - ABC + BAC
\end{aligned}$$

$$\implies \boxed{[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0}$$

Hence, Jacobi Identity is satisfied by the commutator of two square matrices.
Next, we are asked to prove that,

$$[A, BC] = [A, B]C + B[A, C]$$

Proof:

From the given definition of P.B.:

$$[A, BC] = ABC - BCA$$

adding and subtracting BAC

$$\begin{aligned}
\implies &[A, BC] = ABC - BAC - BCA + BAC \\
\implies &[A, BC] = (AB - BA)C - B(CA - AC) \\
\implies &[A, BC] = [A, B]C + B(AC - CA) \\
\implies &\boxed{[A, BC] = [A, B]C + B[A, C]}
\end{aligned}$$

Hence Proved.

Exercises:

9.21.(a) For a one-dimensional system with the Hamiltonian

$$H = \frac{p^2}{2} - \frac{1}{2q^2}$$

show that there is a constant of motion

$$D = \frac{pq}{2} - Ht$$

(b) As a generalization of part (a), for motion in a plane with the Hamiltonian

$$H = |\mathbf{p}|^n - ar^{-n}$$

where p is the vector of the momenta conjugate to the Cartesian coordinates, show that there is a constant of motion

$$D = \frac{\mathbf{p.r}}{n} - Ht$$

(c) The transformation $Q = q$, $p =$ is obviously canonical. However, the same transformation with with t time dilation, $Q = q$, $p = P$, $t = 2t$, is not. Show that, however, the equations of motion for q and p for the Hamiltonian in part**(a)** are invariant under this transformation. The constant of motion D is said to be associated with this invariance.

Sol.9.21.(a)

We are given a Halimtonian,

$$H = \frac{p^2}{2} - \frac{1}{2q^2}$$

We are given a quantity D,

$$D = \frac{pq}{2} - Ht$$

To show that D is a constant of motion we need to show that,

$$\frac{\mathrm{d}D}{\mathrm{d}t} = 0$$

Equation of Motion of D:

$$\begin{aligned}
\frac{\mathrm{d}D}{\mathrm{d}t} &= [D, H] + \frac{\partial D}{\partial t} \\
\implies \frac{\mathrm{d}D}{\mathrm{d}t} &= \left[\frac{pq}{2} - Ht, H\right] - H \\
\implies \frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{1}{2}[pq, H] - t[H, H] - H \\
\implies \frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{1}{2}\left[pq, \frac{p^2}{2} - \frac{1}{2q^2}\right] - 0 - H \\
\implies \frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{1}{4}[pq, p^2] - \frac{1}{4}\left[pq, \frac{1}{q^2}\right] - H \\
\implies \frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{1}{4}\left\{p[q, p^2] + [p, p^2]q - p\left[q, \frac{1}{q^2}\right] - \left[p, \frac{1}{q^2}\right]q\right\} - H \\
\implies \frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{1}{4}\left\{p^2[q, p] + p[q, p]p + 0 - 0 - \frac{1}{q}\left[p, \frac{1}{q}\right]q - \left[p, \frac{1}{q}\right]\frac{q}{q}\right\} - H \\
\implies \frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{1}{4}\left\{p^2 + p^2 - \frac{1}{q^2} - \frac{1}{q^2}\right\} - H
\end{aligned}$$

where we have used

$$\left[p, \frac{1}{q}\right] = \frac{1}{q^2}$$

$$\begin{aligned}
\therefore \frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{p^2}{2} - \frac{1}{2q^2} - H \\
\implies &\boxed{\frac{\mathrm{d}D}{\mathrm{d}t} = 0}
\end{aligned}$$

Therefore, D is a constant of motion.

9.22. Show that the following transformation is canonical by using Poisson Bracket:

$$Q = \sqrt{2q}e^{\alpha}\cos p \qquad\qquad P = \sqrt{2q}e^{-\alpha}\sin p$$

Sol.9.22.
We are given a transformation as follows,

$$Q = \sqrt{2q}e^{\alpha}\cos p$$
$$P = \sqrt{2q}e^{-\alpha}\sin p$$

We know that the fundamental **Poisson Brackets** of the transformed variables have the same value when evaluated with respect to any *canonical* coordinate set. In other words ***the fundamental Poisson Brackets are invariant under canonical transformation.***
Therefore, in order, for the given transformation to be canonical, the Poisson Bracket of Q, P with respect to q and p should be equal to 1.
Using the formula for **Poisson Bracket**,

$$[u,v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i}$$

$$\begin{aligned}
[Q,P]_{q,p} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} \\
\implies [Q,P]_{q,p} &= \frac{e^{\alpha}\cos p}{\sqrt{2q}}\sqrt{2q}e^{-}\alpha\cos p - \frac{e^{-\alpha}\sin p}{\sqrt{2q}}\sqrt{2q}e^{\alpha}(-\sin p) \\
\implies [Q,P]_{q,p} &= \cos^2 p + \sin^2 p \\
\implies [Q,P]_{q,p} &= 1 \\
\therefore \boxed{[Q,P]_{q,p} = [Q,P]_{Q,P}}
\end{aligned}$$

Hence Proved.

9.23. Prove that the following transformation from (q,p) to (Q,P) basis is canonical using Poisson Bracket:

$$Q = \sqrt{2}q\tan p \qquad\qquad P = \sqrt{2}log(\sin p)$$

Sol.9.23.
Note: There seems to be some typo in the question as the Poisson Bracket of (Q,P) with respect to (q,p) doesn't come out to be 1.

9.24. Prove that the transformation defined by $Q = 1/p$ and $P = qp^2$ is canonical using Poisson Bracket.

$$Q = \frac{1}{p} \qquad\qquad P = qp^2$$

Sol.9.24.
We are given a transformation as follows,

$$Q = \frac{1}{p}$$
$$P = qp^2$$

We know that the fundamental **Poisson Brackets** of the transformed variables have the same value when evaluated with respect to any *canonical* coordinate set. In other words ***the fundamental Poisson Brackets are invariant under canonical transformation.***

Therefore, in order, for the given transformation to be canonical, the Poisson Bracket of Q, P with respect to q and p should be equal to 1.
Using the formula for **Poisson Bracket**,

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i}$$

$$\begin{aligned}
[Q, P]_{q,p} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} \\
\implies [Q, P]_{q,p} &= 0 - p^2\left(-\frac{1}{p^2}\right) \\
\implies [Q, P]_{q,p} &= 1 \\
\therefore \boxed{[Q, P]_{q,p} = [Q, P]_{Q,P}}
\end{aligned}$$

Hence Proved.

9.30.(a) Prove that the Poisson Bracket of two constants of motion is itself a constant of motion even when when the constants depend on time explicitly.
(b) Show that if the Hamiltonian and a quantity F are constants of motion, then the nth partial derivative F with respect to t must also be a constant of motion.
(c) As an illustration of this result, consider the uniform motion of a free particle of mass m. The Hamiltonian is certainly conserved, and there exists a constant of motion

$$F = x - \frac{pt}{m}$$

Show by direct computation that the partial derivative of F with t, which is a constant of motion, agrees with $[H, F]$
Sol.9.30.
(a)
Let the two constants of motion be $A(t)$ and $B(t)$. Then since $A(t)$ and $B(t)$ are given to be constants of motion, therefore

$$\begin{aligned}
\frac{\mathrm{d}A(t)}{\mathrm{d}t} &= 0 \\
\implies [A, H] + \frac{\partial A}{\partial t} &= 0 \qquad (4)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}B(t)}{\mathrm{d}t} &= 0 \\
\implies [B, H] + \frac{\partial B}{\partial t} &= 0 \qquad (5)
\end{aligned}$$

The equation of motion of the Poisson Bracket of A and B is given as:

$$\frac{\mathrm{d}[A, B]}{\mathrm{d}t} = [[A, B], H] + \frac{\partial [A, B]}{\partial t} \qquad (6)$$

Lets evaluate the second term in the R.H.S. of (6):

$$\frac{\partial [A, B]}{\partial t} = \left[\frac{\partial A}{\partial t}, B\right] + \left[A, \frac{\partial B}{\partial t}\right] \qquad (7)$$

from (4) and (5) we have

$$\frac{\partial A}{\partial t} = -[A, H]$$
$$\frac{\partial B}{\partial t} = -[B, H]$$

Plugging these back in (7)

$$\frac{\partial [A, B]}{\partial t} = [[H, A], B] + [A, [H, B]]$$

Plugging the above back in (6)

$$\frac{\mathrm{d}[A, B]}{\mathrm{d}t} = [[A, B], H] + [[H, A], B] + [A, [H, B]]$$
$$\implies \frac{\mathrm{d}[A, B]}{\mathrm{d}t} = [H, [B, A]] + [B, [A, H]] + [A, [H, B]]$$

Now using Jacobi's Identity,

$$\implies \frac{\mathrm{d}[A, B]}{\mathrm{d}t} = 0$$

(b) We are given that H is a constant of motion,

$$\therefore \frac{\mathrm{d}H}{\mathrm{d}t} = 0$$

and,

$$\because \frac{\mathrm{d}H}{\mathrm{d}t} = [H, H] + \frac{\partial H}{\partial t}$$
$$\implies \frac{\partial H}{\partial t} = 0 \tag{8}$$

Therefore, H is not a function of time.
We are also given that,

$$\frac{\mathrm{d}F}{\mathrm{d}t} = 0$$
$$\implies [F, H] = -\frac{\partial F}{\partial t}$$
$$\implies \frac{\partial F}{\partial t} = [H, F] \tag{9}$$

Now, lets have a look at the equation of motion of the nth partial derivative of F:

$$\frac{\mathrm{d}\frac{\partial^n F}{\partial t^n}}{\mathrm{d}t} = \left[\frac{\partial^n F}{\partial t^n}, H\right] + \frac{\partial}{\partial t}\left(\frac{\partial^n F}{\partial t^n}\right)$$

using (9)

$$\implies \frac{\mathrm{d}\frac{\partial^n F}{\partial t^n}}{\mathrm{d}t} = \left[\frac{\partial^n F}{\partial t^n}, H\right] + \frac{\partial^n}{\partial t^n}[H, F]$$
$$\implies \frac{\mathrm{d}\frac{\partial^n F}{\partial t^n}}{\mathrm{d}t} = \left[\frac{\partial^n F}{\partial t^n}, H\right] + \left[\frac{\partial^n H}{\partial t^n}, F\right] + \left[H, \frac{\partial^n F}{\partial t^n}\right]$$

the second term on the R.H.S. is clearly zero as H is independent of time as shown earlier already in (8).

$$\implies \frac{\mathrm{d}\frac{\partial^n F}{\partial t^n}}{\mathrm{d}t} = \left[\frac{\partial^n F}{\partial t^n}, H\right] - \left[\frac{\partial^n F}{\partial t^n}, H\right]$$
$$\implies \boxed{\frac{\mathrm{d}\frac{\partial^n F}{\partial t^n}}{\mathrm{d}t} = 0}$$

Therefore, the nth partial derivative of F is a constant of motion.

(c)

$$F = x - \frac{pt}{m}$$
$$\implies \frac{\partial F}{\partial t} = -\frac{p}{m} \tag{10}$$

For a free particle of mass m:

$$H = \frac{p^2}{2m}$$
$$\therefore [H, F] = \left[\frac{p^2}{2m}, x - \frac{pt}{m}\right]$$
$$\implies [H, F] = \left[\frac{p^2}{2m}\right] - \left[\frac{p^2}{2m}, \frac{pt}{m}\right]$$
$$\implies [H, F] = \frac{1}{2m}[p^2, x] - 0$$
$$\implies [H, F] = \frac{1}{2m}\left\{p[p, x] + [p, x]p\right\}$$
$$\implies [H, F] = \frac{1}{2m}(-p - p)$$
$$\implies [H, F] = \frac{-p}{m} \tag{11}$$

from (10) and (11) we have,

$$\therefore \boxed{\frac{\partial F}{\partial t} = [H, F]}$$

Author Notes:

- Despite extreme caution, some errors or typos may have crept in inadvertently. If you find an error or have a correction or suggestion to make then you can either email me or drop a comment on this post on my blog.
- You can check for an updated or revised version of this Solutions Manual on this link.
- You can find solutions to other chapters of Goldstein here.