SOLUTIONS

Chapter 7- The Classical Mechanics of the Special Theory of Relativity

Book: Classical Mechanics 3rd Edition
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December 18, 2016

Exercises:

7.13. Show by direct multiplication of the vector form of the Lorentz transformation equations(Eq.7.9), that

$$r'^2 - c^2t'^2 = r^2 - c^2t^2$$

Sol.7.13. From Eq.7.9. of Goldstein we have,

$$\boxed{\mathbf{r}' = \mathbf{r} + \frac{(\boldsymbol{\beta}.\mathbf{r})\boldsymbol{\beta}(\gamma - 1))}{\beta^2} - \boldsymbol{\beta}\gamma ct}$$

$$\implies r'^2 = \mathbf{r}'.\mathbf{r}'$$

$$\implies r'^2 = \mathbf{r}.\mathbf{r} + \left[\frac{(\boldsymbol{\beta}.\mathbf{r})\boldsymbol{\beta}(\gamma - 1)}{\beta^2}\right]^2 + \gamma^2c^2t^2\boldsymbol{\beta}.\boldsymbol{\beta} + 2\frac{(\boldsymbol{\beta}.\mathbf{r})(\gamma - 1)}{\beta^2}\mathbf{r}.\boldsymbol{\beta} - 2\frac{(\boldsymbol{\beta}.\mathbf{r})(\gamma - 1)(\gamma ct)}{\beta^2}\boldsymbol{\beta}.\boldsymbol{\beta} - 2\gamma ct\mathbf{r}.\boldsymbol{\beta}$$

$$\implies r'^2 = r^2 + \frac{(\boldsymbol{\beta}.\mathbf{r})^2(\gamma - 1)^2}{\beta^2} + \gamma^2c^2t^2\beta^2 + 2(\boldsymbol{\beta}.\mathbf{r})^2\frac{(\gamma - 1)}{\beta^2} - 2(\boldsymbol{\beta}.\mathbf{r})(\gamma - 1)(\gamma ct) - 2\gamma ct\boldsymbol{\beta}.\mathbf{r}.$$

$$\implies r'^2 = r^2 + \frac{(\boldsymbol{\beta}.\mathbf{r})^2(\gamma - 1)^2}{\beta^2} + \gamma^2c^2t^2\beta^2 + 2(\boldsymbol{\beta}.\mathbf{r})^2\frac{(\gamma - 1)}{\beta^2} - 2(\boldsymbol{\beta}.\mathbf{r})\gamma^2ct$$

$$\implies r'^2 = r^2 + (\boldsymbol{\beta}.\mathbf{r})^2\left[\frac{(\gamma - 1)^2}{\beta^2} + 2\frac{(\gamma - 1)}{\beta^2}\right] + \gamma^2c^2t^2\beta^2 - 2(\boldsymbol{\beta}.\mathbf{r})\gamma^2ct \quad (1)$$

Also from Eq.7.9 of Goldstein we have,

$$\boxed{ct' = \gamma(ct - \boldsymbol{\beta}.\mathbf{r})}$$

$$\implies (ct')^2 = \gamma^2(ct - \boldsymbol{\beta}.\mathbf{r})^2$$

$$\implies (ct')^2 = \gamma^2\left(c^2t^2 + (\boldsymbol{\beta}.\mathbf{r})^2 - 2ct\boldsymbol{\beta}.\mathbf{r}\right)$$

$$\implies (ct')^2 = \gamma^2c^2t^2 + \gamma^2(\boldsymbol{\beta}.\mathbf{r})^2 - 2\gamma^2ct(\boldsymbol{\beta}.\mathbf{r}) \quad (2)$$

From (1) and (2) we have,

$$r'^2 - c^2t'^2 = r^2 + (\boldsymbol{\beta}.\mathbf{r})^2\left[\frac{(\gamma - 1)^2}{\beta^2} + 2\frac{(\gamma - 1)}{\beta^2} - \gamma^2\right] + \gamma^2c^2t^2(\beta^2 - 1) - 2(\boldsymbol{\beta}.\mathbf{r})\gamma^2ct + 2\gamma^2ct(\boldsymbol{\beta}.\mathbf{r})$$

$$\because \left[\frac{(\gamma - 1)^2}{\beta^2} + 2\frac{(\gamma - 1)}{\beta^2} - \gamma^2\right] = 0$$

$$\implies r'^2 - c^2t'^2 = r^2 + \gamma^2c^2t^2(\beta^2 - 1)$$

Now, we know that

$$\gamma^2 = \frac{1}{1 - \beta^2}$$

$$\therefore \quad \boxed{r'^2 - c^2t'^2 = r^2 - c^2t^2}$$

Hence Proved.

7.14. Calculate the length of a rod of rest length 2m, moving at 0.73c as observed by an observer at rest. Also, compute at what speed the length in motion of the rod will be half the rest length.

Sol.7.14.

We have, Rest length, $l_0 = 2$m and $\beta = 0.73$. The observed length l is related to the rest length as,

$$l = \frac{l_0}{\gamma}$$

$$\begin{aligned}
&\implies l = l_0\sqrt{1-\beta^2} = 2\sqrt{1-(0.73)^2}\\
&\implies l = 2\sqrt{1-0.5329}\\
&\implies l = 2\sqrt{0.4671}\\
&\implies l = 2(0.6834)\\
&\implies l = 1.3668m
\end{aligned}$$

Now, we are asked to find the speed(or β) for which length is half the rest length, i.e.

$$\begin{aligned}
&\frac{l}{l_0} = \frac{1}{2} = \frac{1}{\gamma}\\
&\implies \gamma = 2\\
&\implies \frac{1}{\sqrt{1-\beta^2}}\\
&\implies 1 = 4(1-\beta^2)\\
&\implies 4\beta^2 = 3\\
&\implies \beta = \sqrt{\frac{3}{4}}\\
&\implies \boxed{v = \sqrt{\frac{3}{4}}c}
\end{aligned}$$

7.18. Calculate the mass of an electron moving at 0.84c. Also, comment whether the electron can be accelerated to the velocity of light. (Rest Mass of an electron is 9.1×10^{-31}kg.)
Sol.7.18. We are given,
Rest mass:

$$m_0 = 9.1 \times 10^{-31}$$

and

$$\beta = 0.84$$

Relativistic mass, m, is given as

$$m = \gamma m_0$$

where m_0 is the rest mass.

$$\begin{aligned}
\implies m &= \frac{1}{\sqrt{1-\beta^2}}m_0\\
&= \frac{1}{\sqrt{1-(0.84)^2}}m_0\\
&= \frac{1}{\sqrt{1-0.7056}}m_0\\
&= \frac{1}{\sqrt{0.2944}}m_0\\
\implies &\boxed{m = 1.8430m_0}
\end{aligned}$$

7.19. A meson of mass m_π at rest disintegrates into a meson of mass m_μ and a neutrino of effectively zero mass. Show that the kinetic energy of the μ meson is

$$T = \frac{(m_\pi - m_\mu)^2}{2m_\pi}c^2$$

Sol.7.19. We are given the following process,

$$\pi \longrightarrow \mu + \nu$$

Four-Momentum of a particle is given by,

$$P^{\mu} = \left[\frac{E}{c}, \mathbf{p}\right]$$

where E is the total energy of the particle,
and $\mathbf{p}$ is the momentum(vector) of the particle.

Using the above formula,
The Four-Momenta of the particles are as follows:
For pion:

$$P^{\alpha}_{\pi} = \left[\frac{E_{\pi}}{c}, \mathbf{0}\right] = [m_{\pi}c, 0]$$

For muon:

$$P^{\alpha}_{\mu} = \left[\frac{E_{\mu}}{c}, \mathbf{p}_{\mu}\right]$$

For neutrino:

$$P^{\alpha}_{\nu} = \left[\frac{E_{\nu}}{c}, \mathbf{p}_{\nu}\right]$$

Conservation of energy and momentum gives:

$$\begin{aligned}
& P^{\alpha}_{\pi} = P^{\alpha}_{\mu} + P^{\alpha}_{\nu} \\
\implies & P^{\alpha}_{\pi} - P^{\alpha}_{\mu} = P^{\alpha}_{\nu} \\
\implies & \left\|P^{\alpha}_{\pi} - P^{\alpha}_{\mu}\right\| = \|P^{\alpha}_{\nu}\| \\
\implies & \left(P^{\alpha}_{\pi} - P^{\alpha}_{\mu}\right)^2 = (P^{\alpha}_{\nu})^2 \\
\implies & P^{\alpha}_{\pi}P_{\alpha\pi} + P^{\alpha}_{\mu}P_{\alpha\mu} - 2P^{\alpha}_{\pi}P_{\alpha\mu} = P^{\alpha}_{\nu}P_{\alpha\nu} \\
\implies & m_{\pi}^2c^2 + m_{\mu}^2c^2 - 2\left[m_{\pi}c\frac{(E_{\mu})}{c} + 0\right] = 0 \\
\implies & 2m_{\pi}E_{\mu} = m_{\pi}^2c^2 + m_{\mu}^2c^2 \\
\implies & E_{\mu} = \frac{(m_{\pi}^2 + m_{\mu}^2)c^2}{2m_{\pi}}
\end{aligned}$$

The above is the total energy(K.E.+Rest Mass Energy) of the muon.
We need to find out the Kinetic Energy of muon, which is the total energy minus the rest mass energy,

$$\begin{aligned}
& T_{\mu} = E_{\mu} - m_{\mu}c^2 \\
\implies & T_{\mu} = \frac{m_{\pi}^2c^2 + m_{\mu}^2c^2 - 2m_{\pi}m_{\mu}c^2}{2m_{\pi}} \\
\implies & \boxed{T_{\mu} = \frac{(m_{\pi} - m_{\mu})^2c^2}{2m_{\pi}}}
\end{aligned}$$

Hence Proved.

7.20. Calculate the mean life of a pion travelling at 0.8c. (Proper mean lifetime for the pion is 28 nano second.)

Sol.7.20.
The proper time τ is related to the dilated time(t) as:

$$\tau = \frac{t}{\gamma}$$
$$\implies t = \gamma\tau$$
$$\implies t = \frac{1}{\sqrt{1-(0.8)^2}}\tau$$
$$\implies t = \frac{1}{\sqrt{0.36}}\tau$$
$$\implies t = \frac{1}{0.6}\tau$$
$$\implies \boxed{t = 1.6667\tau}$$

Therefore the mean lifetime of the travelling pion is 1.6667 times that of the pion at rest.

7.21. A photon may be described classically as a particle of zero mass possessing nevertheless a momentum $h/\lambda = h\nu/$c, and therefore a kinetic energy $h\nu$. If the photon collides with an electron of mass m at rest, it will be scattered at some angle θ with a new energy $h\nu$. Show that the changes in energy is related to the scattering angle by the formula

$$\lambda' - \lambda = 2\lambda_c \sin^2\frac{\theta}{2}$$

where $\lambda_c = h/mc$, is known as the Compton wavelength. Show also that the kinetic of the recoil motion of the electron is

$$T_{e^-} = h\nu\frac{2\left(\frac{\lambda_c}{\lambda}\right)\sin^2\frac{\theta}{2}}{1+2\left(\frac{\lambda_c}{\lambda}\right)\sin^2\frac{\theta}{2}}$$

Sol.7.21.

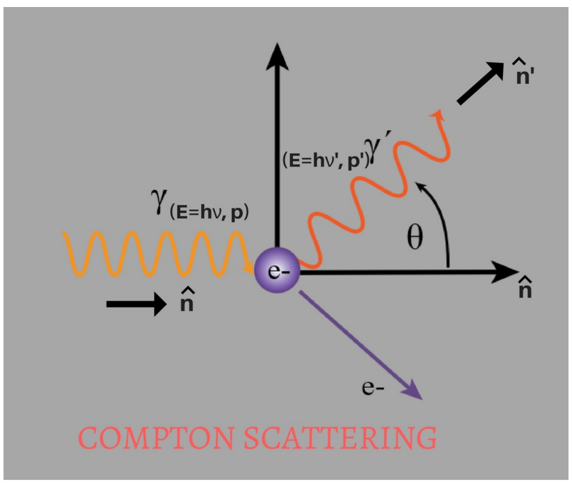

Figure 1: Compton Scattering

The above figure 1 shows the process described in the problem. As you can see it is in fact, the famous, 'Compton Scattering'.
We will solve this problem using 4-momenta.
The four momenta of the particles before and after the collision are as follows:

Photon, before collision, moves with energy $h\nu$ and momentum $\mathbf{p}$ in the direction $\hat{\mathrm{n}}$, therefore its 4-momentum is:

$$P^\mu_\gamma = \left[\frac{E}{c}, \mathbf{p}\right] = \left[\frac{h\nu}{c}, \frac{h\nu}{c}\hat{\mathrm{n}}\right]$$

Photon, after collision, moves with energy $h\nu$ and momentum $\mathbf{p'}$ in the $\hat{\text{n'}}$ direction, therefore its 4-momentum is:

$$P^\mu_{\gamma'} = \left[\frac{E'}{c}, \mathbf{p'}\right] = \left[\frac{h\nu'}{c}, \frac{h\nu'}{c}\hat{\mathrm{n}}'\right]$$

Electron, before collision, is at rest(hence momentum=0) and has rest mass energy mc^2, therefore its 4-momentum is:

$$P^\mu_{e^-} = \left[\frac{E_{e^-}}{c}, 0\right] = [m_e c, 0]$$

Electron, after collision, moves with energy E'_e and momentum $\mathbf{p'}_e$, therefore its 4-momentum is:

$$P^\mu_{e^-{}'} = \left[\frac{E'_{e^-}}{c}, \mathbf{p'_{e^-}}\right]$$

From the conservation of Energy and Momentum we have,

$$\begin{aligned}
& P^\mu_\gamma + P^\mu_{e^-} = P^\mu_{\gamma'} + P^\mu_{e^-{}'} \qquad (3)\\
\implies & P^\mu_\gamma - P^\mu_{\gamma'} + P^\mu_{e^-} = P^\mu_{e^-{}'}\\
\implies & \left\| P^\mu_\gamma - P^\mu_{\gamma'} + P^\mu_{e^-} \right\| = \left\| P^\mu_{e^-{}'} \right\|\\
\implies & P^\mu_\gamma P_{\mu\gamma} + P^\mu_{\gamma'} P_{\mu\gamma'} + P^\mu_{e^-} P_{\mu e^-} - 2P^\mu_\gamma P_{\mu\gamma'} - 2P^\mu_{\gamma'} P_{\mu e^-} + 2P^\mu_\gamma P_{\mu e^-} = P^\mu_{e^-{}'} P_{\mu e^-{}'}\\
\implies & 0 + 0 + m_e^2c^2 - 2\left[\frac{h\nu}{c}\frac{h\nu'}{c} + \frac{h\nu}{c}\frac{h\nu'}{c}\hat{\mathrm{n}}.\hat{\mathrm{n}}'\right] - 2\left[\frac{h\nu'}{c}m_e c + 0\right] + 2\left[\frac{h\nu}{c}m_e c + 0\right] = m_e^2c^2\\
\implies & m_e h(\nu - \nu') = \frac{h\nu h\nu'}{c^2}(1 + \cos\theta)\\
\implies & \nu - \nu' = \frac{h\nu h\nu'}{m_e c^2}(1 + \cos\theta)\\
\implies & \frac{c}{\lambda} - \frac{c}{\lambda} = \frac{hc^2}{\lambda\lambda' m_e c^2}(1 + \cos\theta)\\
\implies & \left(\frac{\lambda' - \lambda}{\lambda\lambda'}\right)c = \frac{c}{\lambda\lambda'}\left(\frac{h}{m_e c}\right)(1 + \cos\theta)\\
\implies & \lambda' - \lambda = \lambda_c\left(2\sin^2\frac{\theta}{2}\right)\\
\implies & \boxed{\lambda' - \lambda = 2\lambda_c\sin^2\frac{\theta}{2}}
\end{aligned}$$

Remarks:
1. If $\theta = 0$, then that isn't much scattering, therefore $\lambda' = \lambda$, as expected.
2. If $\theta = \pi$ (that is backward scattering) and additionally $\lambda' << h/mc$ (that is, $mc^2 << \lambda = E_\gamma$), then $\lambda' = 2h/mc$ so

$$E_\gamma = \frac{hc}{\lambda'} \approx \frac{hc}{\frac{2h}{mc}} = \frac{1}{2}mc^2$$

Therefore, the photon bounces back with an essentially fixed E'_γ, independent of the initial E_γ (as long as E_γ is large enough). This isnt all that obvious.

Now, for the second part of the question, rearranging (3), we get

$$
\begin{aligned}
& P^\mu_\gamma - P^\mu_{\gamma'} = P^\mu_{e^-\prime} - P^\mu_{e^-} \\
\implies & \left(P^\mu_\gamma - P^\mu_{\gamma'}\right)^2 = \left(P^\mu_{e^-\prime} - P^\mu_{e^-}\right)^2 \\
\implies & P^\mu_\gamma P_{\mu\gamma} + P^\mu_{\gamma'} P_{\mu\gamma'} - 2P^\mu_\gamma P_{\mu\gamma'} = P^\mu_{e^-\prime} P_{\mu e^-\prime} + P^\mu_{e^-} P_{\mu e^-} - 2P^\mu_{e^-\prime} P_{\mu e^-} \\
\implies & 0 + 0 + -2\left[\frac{h\nu}{c}\frac{h\nu'}{c} + \frac{h\nu}{c}\frac{h\nu'}{c}\cos\theta\right] = m_e^2c^2 + m_e^2c^2 - 2\left[E'_{e^-} m_e + 0\right] \\
\implies & m_e^2 c^2 + \frac{h\nu h\nu'}{c^2}(1+\cos\theta) = E'_{e^-} m_e \\
\implies & m_e c^2 + \frac{h\nu h\nu'}{m_e c^2}\left(2\sin^2\frac{\theta}{2}\right) = E'_{e^-} \\
\implies & E'_{e^-} = m_e c^2 + 2\frac{h\nu\nu'}{c}\left(\frac{h}{m_e c}\right)\sin^2\frac{\theta}{2}
\end{aligned}
$$

This is the total energy of the recoil electron.
We are interested in the Kinetic Energy, which is the Total Energy minus the Rest mass Energy:

$$
\begin{aligned}
& T_{e^-} = E'_{e^-} - m_e c^2 \\
\implies & T_{e^-} = 2\frac{h\nu}{\lambda'}\lambda_c \sin^2\frac{\theta}{2}
\end{aligned}
$$

using the relation derived in the first part of the question,

$$
\lambda' = \lambda + 2\lambda_c \sin^2\frac{\theta}{2}
$$

Plugging this value of λ' in the expression of Kinetic Energy,

$$
\begin{aligned}
\implies & T_{e^-} = h\nu \frac{2\lambda_c \sin^2\frac{\theta}{2}}{\lambda + 2\lambda_c\sin^2\frac{\theta}{2}} \\
\implies & \boxed{T_{e^-} = h\nu \frac{2\left(\frac{\lambda_c}{\lambda}\right)\sin^2\frac{\theta}{2}}{1 + 2\left(\frac{\lambda_c}{\lambda}\right)\sin^2\frac{\theta}{2}}}
\end{aligned}
$$

which is the required result.

Author Notes:

- Despite extreme caution, some errors or typos may have crept in inadvertently. If you find an error or have a correction or suggestion to make then you can either email me or drop a comment on this post on my blog.
- You can check for an updated or revised version of this Solutions Manual on this link.
- You can find solutions to other chapters of Goldstein here.