2013

# UNIVERSITY OF DELHI

## Department of Physics & Astrophysics

M.Sc. (Physics), I Semester, December 3, 2013

Physics-403 (Electromagnetic Theory & Electrodynamics)

Time: 3 hrs.

Maximum Marks: 70

*Attempt Qs. 1 (which carries 25 marks), and any* **three** *of the rest. (each carrying 15 marks).*

1. *Attempt any* five *parts:*

$5 \times 5 = 25$ marks

(a) A particle of rest mass $m = 1$ GeV and energy 2 GeV collides elastically with another identical particle of the same mass, at rest. The incident particle is scattered by an angle $\theta = \pi/6$. Find the final energy of the incident particle after scattering.

(b) The electromagnetic field tensor is generally expressed as: $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$.

(i) Verify that this is a solution of

$$\partial_\alpha \, {}^\star F^{\alpha\beta} = 0\,, \qquad \text{where} \quad {}^\star F^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu} F_{\mu\nu}\,.$$

(ii) Express $F^{\alpha\beta}\partial_\alpha A_\beta$ in terms of the electric and magnetic fields, $\vec{E}$ and $\vec{B}$ respectively.

(c) Consider a configuration of uniform electric and magnetic fields in a frame $S$:

$$\vec{E} = (E_x, E_y, E_z) = (4, 3, 11)\,, \quad \vec{B} = (B_x, B_y, B_z) = (3, 4, 15)\,.$$

Determine the velocity of a frame $S'$ in which the third components of the transformed fields ($\vec{E}'$ and $\vec{B}'$) in $S'$ are the same as the third components of the fields $E$ and $B$ in the frame $S$, but the projections of $\vec{E}'$ and $\vec{B}'$ on the $(x', y')$-plane are parallel to each other.

(d) Consider a particle of rest mass $m_0$ and charge $q$ moving in a static, uniform electric field $\vec{E}$. The energy of the particle is $\mathcal{E}$. At time $t = 0$, the energy of the particle is $\mathcal{E}_0$ and the velocity is in a direction perpendicular to $\vec{E}$. Show that at a given time $t (> 0)$, the particle has the following velocity components parallel and perpendicular to $\vec{E}$:

$$v_\parallel = \frac{q|\vec{E}|c^2 t}{\sqrt{\mathcal{E}_0^2 + \left(q|\vec{E}|ct\right)^2}} \quad \text{and} \quad v_\perp = \frac{c\sqrt{\mathcal{E}^2 - m_0^2 c^4}}{\sqrt{\mathcal{E}_0^2 + \left(q|\vec{E}|ct\right)^2}}\,.$$

(e) A linear center fed antenna of length $2L$ along the $x$-axis is configured to carry a current:

$$I = I_0\left(1 - \frac{|x|}{L}\right)\sin\omega t\,.$$

Deduce the expression for the strength of an effective oscillating electric dipole for the antenna.

(f) Given a vector potential of the following form:

$$\vec{A} = \vec{C}\frac{e^{ikr}}{r}\left[1 + \mathcal{O}\left(\frac{1}{r}\right) f(\vartheta, \varphi)\right].$$

Show that in the radiation zone, keeping terms up to the leading order in $r^{-1}$,

$$\vec{B} = ik\hat{r} \times \vec{A}; \qquad \vec{E} = \vec{B} \times \hat{r}; \quad \text{and} \quad |\vec{E}| = |\vec{B}|.$$

[Here, $\vec{C}$ is a constant vector specifying the direction of $\vec{A}$, $\hat{r} = \vec{x}/|\vec{x}|$ is a unit vector along the observation direction, and $r = |\vec{x}|$.]

2. (a) For a particle of charge $q$ and rest mass $m_0$, show that the Lorentz force equation generalizes to

$$\frac{dP^\alpha}{d\tau} = \frac{q}{m_0 c} F^{\alpha\beta} P_\beta ,$$

where $\tau$ denotes the charge's proper time, $P^\alpha$ its four-momentum, and $F^{\alpha\beta}$ the electromagnetic field tensor.

5 marks

(b) Show explicitly the following:

(i) $P^\alpha$ and $dP^\alpha/d\tau$ are mutually orthogonal.

(ii) $F^{\alpha\beta}$ is indeed a tensor of rank 2.

1 + 3 marks

(c) The generic second order differential equation satisfied by the electromagnetic four-potential $A^\mu$ is given by

$$\partial_\alpha \partial^\alpha A^\mu - \partial_\alpha \partial^\mu A^\alpha = \frac{4\pi}{c} J^\mu ,$$

where $J^\mu$ is the four-current density vector. Show that in the Coulomb gauge the scalar and vector potentials ($\phi$ and $\vec{A}$ respectively) satisfy:

$$\nabla^2 \phi = -4\pi\rho, \qquad \text{and} \qquad \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{j}_\perp$$

where $\vec{j}_\perp$ is the transverse component of the three-current density $\vec{j}$.

6 marks

3. Find the retarded Green's function for the wave operator ($\partial_\alpha \partial^\alpha \equiv \Box$, or $\Box^2$ in some text) and thus obtain the general retarded solution of the inhomogeneous wave equation satisfied by the electromagnetic 4-potential $A^\mu(\vec{x}, t)$ in the Lorentz gauge, i.e., $\partial_\alpha \partial^\alpha A^\mu = (4\pi/c) J^\mu$. Express your answer in an explicitly covariant form.

12 + 3 marks

4. (a) The power detected per unit solid angle by an accelerated relativistic particle of charge $q$, as a function of the observer's time $t$ is given by

$$\frac{dP(t)}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left[\hat{n} \times \left\{\left(\hat{n} - \vec{\beta}\right) \times \dot{\vec{\beta}}\right\}\right]^2}{\left(1 - \vec{\beta} \cdot \hat{n}\right)^6} ;$$

where $\hat{n}$ is the unit vector directed towards the observer from the emitter, at the emission time $(t')$. Let the acceleration be parallel to the velocity of the particle.

(i) Work out the expression for the power emitted per unit solid angle, as a function of the emission time $t'$.

(ii) Hence show that in the extreme relativistic limit $(\vec{\beta} \to 1)$, the angle $\vartheta$, between $\hat{n}$ and $\vec{\beta}$, for which the emitted power is maximum is half the ratio of the rest energy and the total energy of the particle (i.e. $\mathcal{E}_{\text{rest}}/[2\mathcal{E}_{\text{total}}]$.

3 + 6 marks

(b) A cyclotron operates with a synchronized 1000 Volt potential between its two semi circular disks of 10 meter radius, by rapidly changing the polarity. Assuming Lienard's result for the power emitted by an accelerated charge, determine the maximum energy up to which such a cyclotron can accelerate charged particles.

6 marks

5. (a) Consider a distribution of elementary charges and currents varying harmonically with time.

(i) Show that in the Lorentz gauge the specification of the vector potential $\vec{A}$ alone (derived from the current density $\vec{j}$ alone) is sufficient for a complete determination of the electric and magnetic fields, $\vec{E}$ and $\vec{B}$.

(ii) If, further, the sources (i.e. charges and currents) are localized within a small volume about the origin, then show that the fields $\vec{E}$ and $\vec{B}$ are given by the real parts of

$$e^{i(kr - \omega t)} \times \text{ space-dependent functions} .$$

[Here, $\omega$ is angular frequency of temporal oscillation, $k = \omega/c$, and $r = |\vec{x}|$ is the magnitude of the position vector of the point at which the fields are being detected.]

4 + 4 marks

(b) When the contributions of all other magnetic or electric multipoles are neglected, the electric and magnetic fields due to a temporally oscillating electric dipole of dipole moment $\vec{p}$, in the radiation zone, are given by

$$\vec{E}(\vec{x}, t) = \vec{B}(\vec{x}, t) \times \hat{r} , \quad \text{and} \quad \vec{B}(\vec{x}, t) = k^2 (\hat{r} \times \vec{p}) \frac{e^{ikr}}{r} .$$

Find the time-averaged power flux, and hence the expression for the total power radiated by the oscillating electric dipole as a function $k$ and $\vec{p}$. Verify that if $|\vec{p}| \sim k^{-1}$, then the total power varies as $k^2$.

3 + 3 + 1 marks



$\vec{B}(\vec{x},t) = k^2 (\hat{r} \times \vec{p}) \frac{e^{ikr}}{r} e^{-i\omega t}$