M.Sc. (PHYSICS) / I Sem.

PHY 401 (Classical Mechanics)

Maximum Marks : 70

Time : 3 hours

(Write your Roll No. on the top immediately on receipt of this question paper.)

(Attempt any five questions)

1. (a) A Lagrangian is invariant, to first order in ϵ, under the transformation $q_i \to q_i + \epsilon\, K_i(\{q\})$, where K_i are arbitrary functions of all the q's. Determine the time variation of the quantity Ω, where (3)

$$\Omega = \sum_i \frac{\partial L}{\partial \dot{q}_i} K_i(\{q\}) \ .$$

   (b) Find out the independent invariants for $L = m\,(5\,\dot{x}^2 - 2\,\dot{x}\,\dot{y} + 2\,\dot{y}^2)/2 + c\,(4\,x - 5\,y)$ where c is a constant. (2)

   (c) In a frame F (with coordinates x, y, z), a particle of mass m moves under the potential

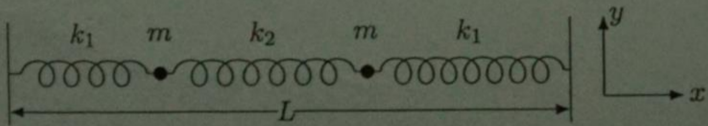
$$V = \frac{m}{2}\left(a_1^2\,x^2 + a_2^2\,y^2 + a_3^2\,z^2\right) \ ,$$

   where a_is are constants. The frame F is, however, rotating, with respect to an inertial frame, with a constant angular velocity $\vec{\omega} = \omega\,\hat{z}$.

   i. *Determine the frequencies* for the normal modes about the point of equilibrium. (4)

   ii. Is there a region in the (ω, a_i) space for which the motion is unstable? If yes, identify the region. If not, why not? (5)

2. (a) Three massless and perfectly elastic springs with zero relaxed length, and particles of mass m each are attached along the x-axis as shown in the figure. The distance between the fixed end points is L. The two outer springs have stiffness k_1, while the third has a stiffness k_2. Neglect gravity.



   i. Find the equilibrium lengths of the three springs. (4)

   ii. If the masses are displaced slightly in the y-direction from the equilibrium positions, find the eigen-frequencies and eigenvectors of the normal modes of small oscillation. (8)

   (b) Construct the independent constants of motion for the Lagrangian $L = m\,(\dot{x}^2 + \dot{y}^2)/2 + k\,(\log x + \log y)$ where k is a constant. (2)

3. (a) Consider the transformation in phase space $(q_i, p_i) \to (Q_i, P_i)$ given by

$$q_1 = \sqrt{Q_1/\omega_1}\,\cos P_1 + \sqrt{Q_2/\omega_2}\,\cos P_2 \qquad q_2 = -\sqrt{Q_1/\omega_1}\,\cos P_1 + \sqrt{Q_2/\omega_2}\,\cos P_2$$

   where ω_is are constants. What must $p_i(\{Q\},\{P\})$ be so that the transformation is canonical? (7)

(b) Consider a particle moving under a central force

$$\vec{F} = -\frac{k}{r^3}\vec{r} - \alpha\vec{r}.$$

Find the condition for the existence of circular orbits. Calculate the time period for a circular orbit of radius R. Calculate the period of radial oscillations for a small perturbation around the circular orbit. (You can assume α to be sufficiently small.) (7)

4. (a) A bead of mass m is constrained to move along a massless rod that is pivoted at the origin and arranged (via an external torque) to rotate with constant angular speed ω in a horizontal plane. A spring with spring constant k and relaxed length zero lies along the rod and connects the mass to the origin.

   i. Find the Lagrangian and the Hamiltonian in terms of the polar coordinate. (2)

   ii. Comment on the conservation of the derived Hamiltonian and energy. (2)

   iii. Find the generating function for a canonical transformation that results in a trivial Hamiltonian. (6)

   (b) Two identical masses m are constrained to move along a horizontal circular hoop. Two identical springs of equal spring constants k, which wrap on the hoop, connect the two masses.

   i. How many normal modes are there of this system? (1)

   ii. Calculate the eigen frequencies and eigen functions of all normal modes. (3)

5. (a) Two thin rods AB and AC of length L each are hinged together at point A. The ends B and C can move along a frictionless horizontal rail. The density of the rods vary as $\rho(x) = \rho_0 \sin(\pi x/L)$ with A being at $x = 0$. *At time $t = 0$, the rods are placed nearly upright with a small* δ between the ends B and C.

   i. Set up the Hamiltonian for the system. (4)

   ii. Determine the subsequent motion of the system. (3)

   iii. Set up the Hamilton-Jacobi equation for the system and solve it. (4)

   (b) In a collection, the density of particles in phase space is $f(\vec{q}, \vec{p}, t)$. When evaluated at the location $(\vec{q}(t), \vec{p}(t))$ of any given particle, f is time-independent. In a circular razor-thin collection, the potential of any particle is given by the function $V(r)$, where r is the radial co-ordinate and ϕ the angular coordinate. For a particle of mass m, calculate χ where (3)

$$\chi \equiv m\frac{\partial f}{\partial t} + p_r\frac{\partial f}{\partial r} + \frac{p_\phi}{r^2}\frac{\partial f}{\partial \phi} + \left[\frac{p_\phi^2}{r^3} - V'(r)\right]\frac{\partial f}{\partial p_\phi}$$

6. A particle of mass m and charge q moves in the equatorial ($\theta = \pi/2$) plane of a magnetic dipole μ. It is given that the dipole's vector potential is

$$\vec{A} = \frac{\mu \sin\theta}{4\pi r^2}\hat{\phi}$$

where (r, θ, ϕ) denote the spherical polar coordinates.

(a) Determine the Hamiltonian for the particle. (*It might be smarter to do it from first principles.*) (5)

(b) Starting from infinity, at time $t = 0$ and with an initial speed $v(0)$, the particle approaches the dipole with an impact parameter b. Determine the instantaneous speed $v(t)$. (2)

(c) For $q\mu > 0$, evaluate the distance of closest approach D_0. How does D_0 depend on the initial angular velocity $\dot{\phi}(0)$? (7)