Your roll no. ....

Department of Physics and Astrophysics, University of Delhi
M.Sc.(Physics) Semester I   Final examination

PHY401 (Classical Mechanics)   *November 23, 2013*

Time : 3 hours.   *Marks : 70*

Attempt any five questions.

1. (a) A point particle of mass m is constrained to move on a plane with a constant speed v. Given that it has to start from (x_1, y_1) and reach the point (x_2, y_2), find the curve of motion, using calculus of variations, that will minimize the travel time. (2)

(b) A point particle moves in a central force field in three dimensions described by some potential energy function $V(r)$. Show that the trajectory of the particle necessarily has to lie in a plane. Show that the areal velocity of the particle is constant. (3)

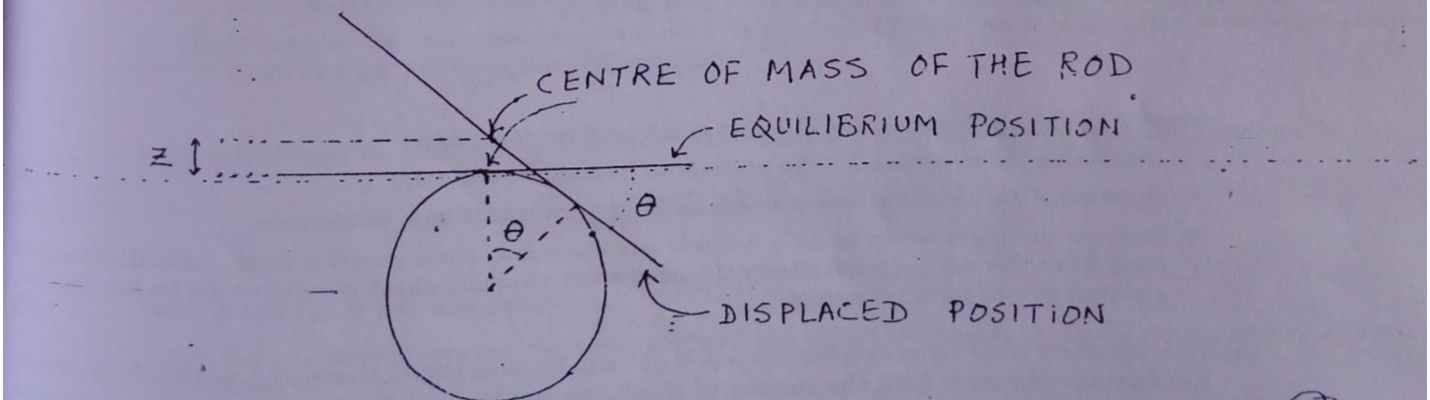
(c) The dynamics of a classical system with N degrees of freedom is described by a Hamiltonian $H(q,p,t)$ where $q = (q_1, q_2, ..., q_N)$ and $p = (p_1, p_2, ..., p_N)$. For a dynamical variable $F(q,p,t)$, prove using Hamilton's equations of motion that as the system evolves with time, the rate of change of F is given by (2)

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}.$$

(d) A projectile with mass m, angular momentum l and energy E comes from far away and is incident on a fixed target particle sitting at $r = 0$. Given that the potential energy of the projectile at a distance r from the target is k/r^2, calculate the impact parameter and the distance of closest approach to the target particle. (k is a constant.) (3)

(e) The velocity $\vec{v}$ (with respect to an inertial frame K) of any point P fixed in a rigid body is given by $\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$, where $\vec{V}$ is the translational velocity (with respect to K) of a point O fixed in the rigid body, $\vec{r}$ is the position vector of P with respect to O and $\vec{\Omega}$ is the angular velocity of the rigid body about O. Show that $\vec{\Omega}$ is independent of the choice of the point O in the rigid body. (4)

2. A thin uniform rod of length ℓ and mass M is balanced horizontally on a fixed cylinder of radius of cross-section R as shown in the figure, such that the centre of mass of the rod is resting on the cylinder and the rod is perpendicular to the axis of the cylinder. Assume that there is no friction between the rod and the cylinder. The force of gravity acts vertically downwards (g = acceleration due to gravity).



(a) Calculate the moment of inertia I of the rod about its centre of mass. (3)

(b) If one end of the rod is lowered by a very small amount and then released, it is seen that the rod performs an oscillatory motion without slipping in a vertical plane perpendicular to the axis of cylinder. How many degrees of freedom are needed to describe this motion? Explain (in two sentences at maximum) what gives rise to the restoring force that leads to the oscillatory motion. (3)

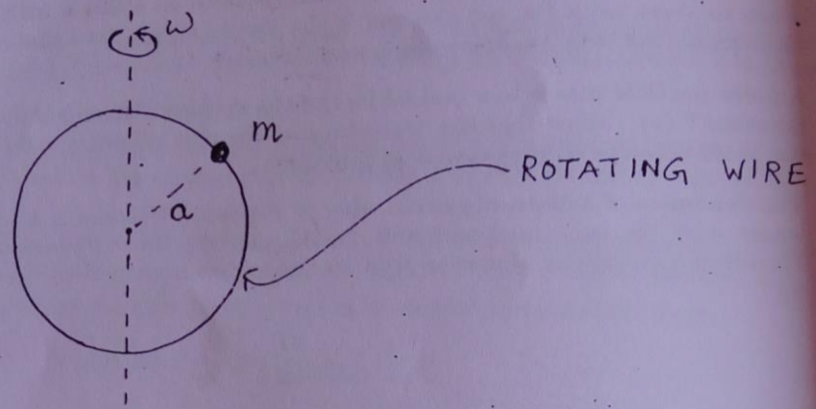
(c) For small angle oscillations of the rod, the Lagrangian is given by

$$L = \frac{1}{2} M \dot{z}^2 + \frac{1}{2} I \dot{\theta}^2 - M g z.$$

Here z is the vertical displacement of the centre of mass from its equilibrium position and θ is the angular displacement of the rod from the horizontal position. For sufficiently small oscillations, $z = R\theta^2/2$. Using this, find the frequency of oscillation as a function of g, ℓ and R. (4)

(d) Prove the relation $z = R\theta^2/2$ that you used in part (c). (4)

3. (a) A tiny bead of mass m is allowed to slide without friction along a very thin but rigid circular wire (of radius a) that is rotating with a constant angular frequency ω about the vertical axis passing through its centre as shown in the figure. The force of gravity mg acts vertically downwards.



i. Write down the constraints for the bead using spherical polar coordinates. How many degrees of freedom are there in this problem? (2)

ii. Formulate the Lagrangian for the bead. What, if any, are the conserved quantities? (3)

iii. Obtain the equation(s) of motion for the bead. (2)

iv. Find a solution corresponding to the beam *not* sliding along the wire at all. Show that such a solution exists only if ω is greater than a critical value ω_c. Determine ω_c. (4)

(b) A particle of mass m carrying charge q is constrained to move on the xy-plane. An external magnetic field $\vec{B} = B_0\,\hat{k}$, (where B_0 is constant and uniform), is applied. Calculate the Poisson bracket $\{\dot{x}, \dot{y}\}$. *(You may use $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$.)* (3)

4. (a) Consider a particle of mass m in a force field given by

$$\vec{F}(\vec{r}) = \frac{-k}{r^2}\, e^{-ar}\, \hat{r}\,,$$

where $\hat{r}$ is the unit vector along $\vec{r}$, k and a are positive constants.

i. Determine the condition for the particle to have a circular orbit. (3)

ii. Determine the condition under which such an orbit would be a stable one. (3)

iii. Suppose the particle was in such a stable circular orbit. Now, give it a small radial displacement away from this orbit. Qualitatively describe what the subsequent motion would be like and give a sketch of a possible trajectory. (2)

(b) The spacecraft of the Mars Orbiter Mission is given to be in an elliptical orbit around Earth. Its closest and farthest distances from the surface of Earth are 250 km and 80000 km, respectively. Suppose the booster rockets are fired to increase the farthest distance to 8×10^5 km but keeping the closest distance the same (250 km).

i. Calculate the change in the eccentricity of the orbit. (3)

ii. Has the orbital energy of the spacecraft increased or decreased after the rockets were fired? Give reasons. (1.5)

iii. Has the orbital angular momentum increased or decreased in the process? Give reasons. (1.5)

Hint: You may use $r(\theta) = (l^2/mk)\,(1 + \epsilon\cos\theta)^{-1}$, where the symbols have their usual meanings. The eccentricity is $\epsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$. The Earth is a sphere of radius of about 6000 km. The mass of the spacecraft may be assumed to remain the same after firing the booster rockets.

5. Consider the Hamiltonian $H(q,p) = p^2/(2m) + m\omega^2 q^2/2$ for a single degree of freedom.

(a) Show that

$$S(t,q;\alpha) = \int_0^q d\bar{q}\,\sqrt{2m\alpha - m^2\omega^2\bar{q}^2} - \alpha t,$$

with α being a constant, is a solution to the Hamilton-Jacobi equation for this system. (2)

(b) Show that the solution $S(t,q;\alpha)$ in part (a) is related to the generating function of a canonical transformation $(q,p) \to (Q,P) \equiv (\beta,\alpha)$ which is such that the Hamiltonian H' for the new variables vanishes. Express p and Q in terms of the appropriate derivatives of $S(t,q;\alpha)$. (3)

(c) Find the solution for $q(t)$ and $p(t)$ using the results of part (b). (6)

(d) If at time $t=0$, $q(0)=A$ and $p(0)=0$, find α and β. What are the physical meanings of A and α? (3)

6. (a) A system in three dimensions is characterized by the Lagrangian

$$L = \frac{m}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - \frac{k}{\sqrt{x^2+y^2+z^2}} + \sigma J_z,$$

where k, σ are constants and $\vec{J} = \vec{r}\times m\dot{\vec{r}}$.

i. Starting from this Lagrangian, determine the Euler-Lagrange equations of motion for the system. (1.5)

ii. Find the canonical momenta p_x, p_y and p_z. (1.5)

iii. Determine the Hamiltonian for the system. (2)

iv. Calculate the Poisson brackets : (3)

$$\{\dot{x},\dot{y}\},\quad \{\dot{x},\dot{z}\},\quad \{\dot{y},\dot{z}\}.$$

v. Identify the independent constants of motion. (2)

(b) A system is governed by the Lagrangian

$$L = \frac{m}{2}\dot{q}^2 - \frac{k}{2}q^2 + a\,q\,\dot{q},$$

where m, k and a are constants.

i. Find the Hamiltonian for this system. (1)

ii. Determine the canonical transformation $(q,p)\to(Q,P)$ that leaves the transformed Hamiltonian to be that for a simple harmonic oscillator. (3)

7. A particle of mass m, energy E and angular momentum J is scattered by a potential $V(r)$ given by

$$V(r) = \begin{cases} -V_0 & r \le a \\ 0 & r > a \end{cases},$$

where a and V_0 are positive constants.

(a) Is the interaction between the projectile and the target an attractive one or a repulsive one? Give reasons. (1)

(b) Starting from the expressions for energy and angular momentum, show that the angle of scattering χ is given by

$$\chi = \left|\pi - 2b\int_{r_{\min}}^{\infty}\frac{dr}{r^2}\left[1 - \frac{V(r)}{E} - \frac{b^2}{r^2}\right]^{-1/2}\right|$$

where b is the impact parameter and $r_{\min}$ is the distance of closest approach to $r=0$. (4)

(c) If $J \le a\sqrt{2mE}$, find the differential scattering cross section. (4)

(d) Will the differential cross section change if V_0 is negative? Explain. (2)

(e) If $J > a\sqrt{2mE}$, how will the scattering cross section change? (3)