This question paper contains 4 printed pages]

Your Roll No................

9653A

B.A./B.Sc. (Hons.)/I **B**

MATHEMATICS-Unit-IV

(Analysis-II)

(Admissions of 2008 and before)

Time : 2 *Hours* *Maximum Marks* : 38

(*Write your Roll No. on the top immediately on receipt of this question paper.*)

All questions are compulsory.

Attempt any *two* parts from each question.

1\. (*a*) Show that the function f defined on $\mathbf{R}$ by :

$$f(x) = \begin{cases} 4x & , \text{ when } x \text{ is irrational} \\ 5x & , \text{ when } x \text{ is rational} \end{cases}$$

is continuous only at $x = 0$.

(*b*) Prove that the image of a closed interval under a continuous function is a closed interval.

(c) What do you understand by a uniformly continuous function ? Let $f(x)$ be a real valued function on the closed and bounded interval I. Show that f is continuous on I if and only if f is uniformly continuous on I. 5+5

2. (a) If f is a given function on $[a, b]$ such that f' is defined on $[a, b]$ and $|f''(x)| \leq M$ for all $x \in [a, b]$, then show that :

$$\left|f(b) - f(a) - \frac{1}{2}(b-a)\{f'(b) + f'(a)\}\right| \leq \frac{1}{2} M (b-a)^2 .$$

(b) Obtain the Maclaurin series expansion of :

$$f(x) = \log(1+x) \text{ for } |x| < 1 .$$

(c) (i) Let f be a function with domain D which contains 0 and let g be the function defined on D by setting :

$$g(x) = x\, f(x), \text{ for all } x \in D.$$

Prove that if f be continuous at $x = 0$, then g is derivable at $x = 0$.

(*ii*) If f and g are both defined and continuous on $[a, b]$ and are derivable on $]a, b[$ and if $f'(x) = g'(x)$ for all x in $]a, b[$, then prove that $f(x)$ and $g(x)$ differ only by a constant on $[a, b]$. 4½+4½

3. (*a*) If $0 < x < 1$, show that :

$$2x < \log\left[\frac{1+x}{1-x}\right]$$

Deduce that :

$$e < \left(1 + \frac{1}{n}\right)^{n+1/2}$$

(*b*) Evaluate :

(*i*) $\lim\limits_{x \to 0+} (\cot x)^{\frac{1}{\log x}}$

(*ii*) $\lim\limits_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$

(*c*) Show that the function f, defined by :

$$f(x) = x^p (1-x)^q \ \forall \ x \in \mathbf{R}$$

where p and q are positive integers, has a maximum value, whatever the values of p and q may be. 4½+4½

4. (*a*) Evaluate any *two* of the following integrals :

(*i*) $\int_0^1 x \sqrt{\frac{1 - x^2}{1 + x^2}}\, dx$

(*ii*) $\int \frac{dx}{(x + 1)\sqrt{2x^2 + 3x + 4}}$

(*iii*) $\int \frac{dx}{\sqrt{1 + x} + \sqrt[3]{1 + x}}$

(*b*) If :

$$I_n = \int_0^{\pi/2} x \cos^n x\, dx, \text{ and } n > 1,$$

prove that :

$$I_n = \frac{n - 1}{n} I_{n-2} - \frac{1}{n^2}$$

Deduce that :

$$I_5 = \frac{4\pi}{15} - \frac{149}{225}$$

(*c*) Find the entire length of the asteroid :

$$x^{2/3} + y^{2/3} = a^{2/3}$$

Or

Find the surface area of the solid generated by revolving the curve :

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$$

about x-axis. 5+5