[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 6604 **D** **Your Roll No...............**

Unique Paper Code : 235104

Name of the Course : **B.Sc. (Hons.) Mathematics – I**

Name of the Paper : Algebra – I

Semester : I

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All **six** questions are compulsory.
3. Do any **two** parts from each question.

1. (a) Suppose a complex polynomial p(t) can be factored in two ways as

$$p(t) = \prod_{i=1}^{m}(t - a_i) = \prod_{j=1}^{n}(t - b_j)$$

Show that m = n and that the a_i's are the same as the b_j's in some order. (6)

(b) Use Descarte's rule of signs to verify that $t^{11} + t^8 - 3t^5 + t^4 + t^3 - 2t^2 + t - 2$ has at most 5 positive and two negative zeroes. Deduce that it has at least 4 nonreal zeroes. (6)

(c) (i) Consider the polynomial equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

Prove that the product of two of its roots is equal to the product of the other two if and only if $r^2 = p^2s$.

(ii) Find the polar representation of the complex number $z = -1 + i\sqrt{3}$ and write its extended argument. (4,2)

2. (a) (i) Suppose $z_1 = 1 + i$ and $z_2 = -1 + i$. Find measures of angles M_1OM_2 and M_2OM_1, where M_1 represents z_1 and M_2 represents z_2.

(ii) Let z_1, z_2, z_3 be the coordinates of the vertices A_1, A_2, A_3 of a positively oriented triangle, prove that triangle $A_1A_2A_3$ is an equilateral triangle if and only if

$z_3 - z_1 = \varepsilon (z_2 - z_1)$, where $\varepsilon = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$. (2.5,4)

(b) Solve the equation $z^6 + iz^3 + i - 1 = 0$. (6.5)

(c) Find $|z|$ and arg z for

$$z = \frac{(-1+i)^4}{(\sqrt{3}-i)^{10}} + \frac{1}{(2\sqrt{3}+2i)^4}$$ (6.5)

3. (a) State second principle of mathematical induction and use it to prove that every integer greater than 1 is either a prime or a product of primes. (5)

(b) (i) For $a, b \in \mathbb{R}$, define $a \sim b$ if and only if $a - b \in \mathbb{Z}$.

(I) Prove that $\sim$ defines an equivalence relation on $\mathbb{R}$.

(II) Find the equivalence classes of 5 and ½.

(ii) Determine whether the following relation R is a function with domain {1,2,3,4}

$R = \{(1,1), (2,1), (3,1), (4,1), (3, 3)\}$. (4,1)

(c) Define a function $g : \mathbb{Z} \to \mathbb{Z}$ by $g(x) = 2x^2 + 7x$. Is g one to one and / or onto ? Explain. (5)

4. (a) Show that the function $f : A \to \mathbb{R}$ given by $f(x) = 1 + \frac{1}{x-4}$ is one to one, where $A = \{x \in \mathbb{R} / x \neq 4\}$. Find the range of f and a suitable inverse. (5)

(b) (i) Prove that the intervals (−1, 2) and (4, 6) have the same cardinality.

(ii) Prove or disprove : If A $\subsetneq$ B, then A and B do not have the same cardinality. (4,1)

(c) (i) Find the gcd of 630 and 196 using division algorithm method and hence write gcd (630, 196) as a linear combination of the two numbers.

(ii) Does $7 \in [-13] \pmod 5$? Give reasons. (4,1)

5. (a) Let $A = \begin{pmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ -7 \\ -3 \end{pmatrix}$. Determine if b is a linear combination of the vectors formed from the columns of matrix A. (7.5)

(b) Balance the following chemical equation by solving a suitable system of linear equations,

$$C_3H_8 + O_2 \rightarrow CO_2 + H_2O,$$

That is when propane gas burns , propane (C_3H_8) combines with oxygen (O_2) to form carbon dioxide (CO_2) with water (H_2O). (7.5)

(c) (i) Determine by inspection whether the following vectors are linearly independent. Justify your answer. Where $v_1 = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} -6 \\ 5 \\ 4 \end{pmatrix}$.

(ii) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation such that $T(x_1, x_2) = (x_1 + x_2, 4x_1 + 5x_2)$. Find x such that $T(x) = (3, 8)$. (3,4.5)

6. (a) (i) Find the inverse of the matrix A, if it exists, where

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}$$

using E-Row operation method.

(ii) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a vertical shear transformation that maps e_1 into $e_1 - 2e_2$ but leaves the vector e_2 unchanged. Find the standard matrix of T. (4.5,3)

(b) (i) The vector $x = \begin{pmatrix} -7 \\ 5 \end{pmatrix}$ is in a subspace of H with the basis $B = \{b_1, b_2\}$, where $b_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $b_2 = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$. Find the B-coordinate vector of x.

(ii) Suppose a 3×5 matrix A has three pivot columns. Is Col A $= \mathbb{R}^3$? Is Nul A $= \mathbb{R}^2$? Explain your answer. (3.5,4)

(c) Define Nul A and Col A for a matrix A. Find a basis and dimension for the column space of the matrix $A = \begin{pmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{pmatrix}$. (7.5)