[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 6603 **D** **Your Roll No...............**

Unique Paper Code : 235103

Name of the Course : **B.Sc. (Hons.) Mathematics**

Name of the Paper : Analysis – I / Code : MAHT 102

Semester : I

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **three** parts from each question.
4. **All** question carry equal marks.

1. (a) Let S be a bounded subset of $\mathfrak{R}$. For some scalar $\lambda \in \mathfrak{R}$, define $\lambda S = \{\lambda s : s \in S\}$. If $\lambda < 0$, prove that $\sup(\lambda S) = \lambda$, inf S and $\inf(\lambda S) = \lambda$ sup S.

   (b) Show that the union of an arbitrary family of open sets is an open set. What can you say about the intersection of an arbitrary family of open sets ? Justify your answer.

   (c) If $S \subseteq \mathfrak{R}$ is a nonempty bounded set and $I = [\inf S, \sup S]$, show that $S \subseteq I$. Moreover, if J is any closed and bounded interval containing S, show that $I \subseteq J$.

   (d) State and prove order completeness property of $\mathfrak{R}$.

2. (a) Define limit of a convergent sequence. Test the convergence of the following sequences by applying the definition :

(i) $\lim \dfrac{n^2 - 1}{2n^2 + 3}$

(ii) $\lim \dfrac{4n^3 + 3n}{n^3 - 6}$

(b) If $<x_n>$ is a bounded sequence and $\lim_{n\to\infty} y_n = 0$ then show that $\lim_{n\to\infty} x_n y_n = 0$.

(c) Prove that $\lim_{n\to\infty} n^{1/n} = 1$.

(d) Let $<x_n>$ be a convergent sequence and $<y_n>$ is such that for any $\varepsilon > 0$ there exists M such that $|x_n - y_n| < \varepsilon$ for all $n \geq M$. Does it follow that $<y_n>$ is convergent ? Justify your answer.

3. (a) Let $x_1 \geq 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$ for $n \in N$. Show that $<x_n>$ is decreasing and bounded below by 2. Find the limit.

(b) Let $x_n = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots\ldots + \dfrac{1}{n^2}$ for each $n \in N$. Prove that $<x_n>$ is increasing and bounded and hence converges.

(c) State and prove Bolzano-Weierastrass Theorem for sequences.

(d) Consider the following sequences :

$$x_n = \cos\frac{n\pi}{3} \qquad y_n = \frac{3n}{4n+1} \qquad z_n = \left(\frac{-1}{2}\right)^n$$

(i) In each of the above sequences find a monotone subsequence.

(ii) Which sequence of the above three sequences are convergent ?

4. (a) Let $<x_n>$ be a sequence of real numbers. Prove that if lim inf x_n = lim sup $x_n = l$ where l is a finite real number, then lim x_n is defined and lim inf x_n = lim sup x_n = lim $x_n = l$.

(b) Find the limit superior and limit inferior of the following sequences.

$$x_n = (-2)^n \qquad y_n = \sin\frac{n\pi}{3} \qquad z_n = 1 + \frac{(-1)^n}{n}$$

(c) Define Cauchy sequence. If $x_n = \sqrt{n}$, show that $<x_n>$ satisfies $\lim |x_{n+1} - x_n| = 0$, but it is not a Cauchy sequence.

(d) Show that a bounded monotone increasing sequence is a Cauchy sequence.

5. (a) Show that if $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers, then $\sum \sqrt{a_n b_n}$ converges.

(b) Test the convergence of **any two** of the following :

(i) $\sum \frac{2 + \cos n}{3^n}$

(ii) $\sum_{n=1}^{\infty} \frac{n^5}{(n+1)!}$

(iii) $\sum_{n=0}^{\infty} 2^{((-1)^n - 1)}$

(c) Discuss the convergence of the series $\sum \frac{1}{n^p}$ where $p > 0$.

P.T.O.

(d) Define an alternating series ? State the result which can be used to discuss convergence of such series and test the convergence of the following series.

$$\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{7}}+\frac{1}{\sqrt{9}}\ldots\ldots$$