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Your Roll No.....................................

1462

B.Sc. (Hons.)/I A

MATHEMATICS—Paper III

(Algebra-I)

(Admission of 2009 and onwards)

Time : 3 *Hours* *Maximum Marks* : 75

(Write your Roll No. on the top immediately on receipt of this question paper.)

All questions are compulsory.

Do any *two* parts from each question.

1. (*a*) The diagonals of a square intersect at A and B is one vertex of the square. Prove that the four vertices of the square are represented by the numbers :

$a + (b - a),\ a + i(b - a),\ a - (b - a),\ a - i(b - a),$

where the points A, B are represented by the complex numbers a, b. 6

P.T.O.

(*b*) Determine the values of the constants a, b, c in the equation :

$$2^4 \cos^5 \theta = a \cos 5\theta + b \cos 3\theta + c \cos \theta.$$ 6

(*c*) Using Descartes' rule of signs, show that the equation :

$$x^6 - 2x^5 + x^4 + x^2 - 2x + 1 = 0$$

must have at least two complex roots. 6

2. (*a*) Define congruence modulo n, where n is a positive integer. Prove that it is an equivalence relation. Find all congruence classes of integers modulo 5. 5

(*b*) Let $A = \{x \in \mathbf{R} | x \neq 2\}$ and $B = \{x \in \mathbf{R} | x \neq 1\}$. Define $f : A \rightarrow B$ and $g : B \rightarrow A$ by :

$$f(x) = \frac{x}{x-2}, \; g(x) = \frac{2x}{x-1}$$

(*i*) Find $(f \circ g)(x)$.

(*ii*) Are f and g invertible ? If so, find their inverses. 5

(c) Define cardinality of a set. Find the cardinalities of the following sets :

(i) $2\mathbf{Z}$, the set of all even integers.

(ii) $\mathbf{N} \times \mathbf{N}$, where $\mathbf{N}$ is the set of natural numbers. 5

3. (a) (i) Find all the elements of order 2 in the dihedral group D_4.

(ii) Prove that a group of order 3 is cyclic.

(iii) Give an example of a group of order 21. 7½

(b) (i) Find the inverse of the element $\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$ in $GL(2, \mathbf{Z}_{11})$.

(ii) Let H be a subgroup of $\mathbf{R}$ under addition and let $K = \{2^a | a \in H\}$. Prove that K is a subgroup of $\mathbf{R} - \{0\}$ under multiplication. 7½ (*c*) Define order of an element of a group. Let G be a group and let $a \in$ G. If a has infinite order, then all district powers of a are distinct group elements. If a has finite order, say n, then prove that :

$$\langle a \rangle = \{e, a, a^2, \ldots\ldots\ldots\ldots, a^{n-1}\} \text{ and}$$

$a^i = a^j$ if and only if n divides $i - j$. 7½

4. (*a*) Give an example of a non-commutative ring with 16 elements. 4

(*b*) Let R be a commutative ring and

$S = \{a \in R : a^n = 0 \text{ for some positive integer } n\}$,

show that S is a subring of R. 4

(*c*) Prove that a finite integral domain is a field. Find all zero divisors in $Z_3 \oplus Z_6$. 4

5. (*a*) Find the general solution of the system whose augmented matrix is :

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}.$$ 7½

(*b*) (*i*) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation such that :

$$T(u) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } T(v) = \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$

where $u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Find $T(3u + 2v)$.

(*ii*) Do the vectors :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

span $\mathbf{R}^4$? Why or why not ? Are they linearly independent ? Justify. 7½ (*c*) Determine the rank of the matrix :

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

by reducing it to the row-echelon form. 7½

6. (*a*) Define $T : \mathbf{P}_2 \to \mathbf{R}^2$ by :

$$T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix},$$

where

$$\mathbf{P}_2 = \{a + bx + cx^2 | a, b, c \in \mathbf{R}\}$$

(*i*) Show that T is a linear transformation.

(*ii*) Find a polynomial p for $\mathbf{P}_2$ that spans the kernel of T and describe the range of T. 7½

(*b*) Find $[x]_{\mathbf{B}}$, the co-ordinate matrix of x relative to $\mathbf{B}$, where :

$$\mathbf{B} = \left\{\begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix}\right\} \text{ and } x = \begin{bmatrix} 2 \\ -6 \end{bmatrix}.$$

Also show that $\mathbf{B}$ is a basis for $\mathbf{R}^2$. 7½ (*c*) Find the characteristic equation of :

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Also find the eigenvalues with multiplicities. 7½