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Your Roll No................

1461

B.Sc. (Hons.)/I **A**

MATHEMATICS—Paper II

(Analysis—I)

(Admissions of 2009 and onwards)

Time : 3 Hours *Maximum Marks : 75*

(Write your Roll No. on the top immediately on receipt of this question paper.)

All the questions are compulsory.

Use of basic Calculator is allowed.

1. Attempt any *two* parts :

(*a*) (*i*) If $a, b \in \mathbf{R}$, show that $|a + b| = |a| + |b|$ if and only if $a\,b \geq 0$. 3

(*ii*) Let S be a non-empty subset of **R** that is bounded above and let '*a*' be any new number. Prove that : 2½

$$\sup(a + S) = a + \sup S$$

where

$$a + S = \{a + s : s \in S\}.$$

(*b*) (*i*) If $x \in \mathbf{R}$, show that there exists $n \in \mathbf{N}$ such that $n > x$. 3

(*ii*) Given any $x \in \mathbf{R}$, show that there exists a unique $n \in \mathbf{Z}$ such that $n - 1 \leq x < n$. 2½

(*c*) (*i*) If $I_n = [a_n, b_n]$, $n \in \mathbf{N}$ is a nested sequence of closed and bounded intervals in $\mathbf{R}$, then show that there exists a $\xi \in \mathbf{R}$ such that : 3

$$\xi \in I_n, \forall n \in \mathbf{N}.$$

(*ii*) Define limit point of a set in $\mathbf{R}$. Determine the set of all limit points of the set of all integers. 2½

2. Attempt any *three* parts :

(*a*) Let (r_n) be a sequence of positive real numbers such that : 5

$$\lim \left(\frac{x_{n+1}}{x_n} \right) = L \text{ exists in } \mathbf{R}.$$

If L < 1, prove that :

$$\lim (x_n) = 0.$$

What happens when L = 1 ? Justify your answer.

(b) Determine the following limits and also state all theorems used to evaluate these limits : 5

(i) $\lim (3\sqrt{n})^{1/2w}$

(ii) $\lim (n!^{1/n^2})$.

(c) (i) If $X = (x_n)$ is a sequence of real numbers, show that there is a subsequence of X that is monotone. 3

(ii) Suppose that $x_n \geq 0, \forall\ n \in \mathbf{N}$ and that $\lim \left((-1)^n x_n\right)$ exists in **R**. Show that (x_n) converges. 2

(d) (i) Show that every Cauchy sequence in **R** is convergent. 3

(ii) If $x_n = \sqrt{n}, n \in \mathbf{N}$, show that (x_n) satisfies $\lim (|x_{n+1} - x_n|) = 0$ but that it is not a Cauchy sequence. 2

3. Attempt any *two* parts :

(*a*) Let Σa_k and Σb_k be two infinite series of positive real numbers such that :

$$\lim\left(\frac{a_k}{b_k}\right) = 0.$$

If Σb_k is convergent prove that Σa_k is also convergent. Hence or otherwise prove that $\Sigma (-1)^k \frac{\log k}{k^2}$ is convergent. 4,2

(*b*) Examine for the convergence, conditionally convergence and absolute convergence of the following series :

(*i*) $\sum_{2}^{\infty} (-1)^k \frac{1}{k \log k}$

(*ii*) $\Sigma(-1)^k \frac{1}{k^p}, p \in \mathbf{R}$ 3,3

(*c*) (*i*) Let Σa_k be a convergent series in **R**. Prove that $\lim (a_k) = 0$. 2

(*ii*) Examine for the convergence of the series : 4

$$\Sigma\frac{2^k + k}{3^k - k} \text{ and } \Sigma\frac{\sin n\alpha + \cos^2 n\alpha}{n^2}, \alpha \in \mathbf{R}.$$

4. Attempt any *three* parts :

(*a*) (*i*) Determine G condition on $|x - 4|$ that will assure that :

$$|\sqrt{x} - 2| < \frac{1}{2}.$$

(*ii*) Use the definition of limit to show that :

$$\lim_{x \to 2} (x^2 + 4x) = 12.$$ 5

(*b*) Prove that :

$$\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$$

does not exist but that :

$$\lim_{x \to 0} x \cos \frac{1}{x} = 0.$$ 5

(*c*) (*i*) Let $A \subseteq \mathbf{R}$, $f : A \to \mathbf{R}$ and $C \in A$. If 'f' is continuous at C, show that for every sequence (x_n) in A that converges to C, the sequence $(f(x_n))$ converges to $f(C)$.

(*ii*) Let $f : \mathbf{R} \to \mathbf{R}$ be defined as :

$$f(x) = \begin{cases} 1 & \text{if } x \in Q \\ -1 & \text{if } x \in \mathbf{R}, Q \end{cases}$$

Show that 'f' is discontinuous at every $C \in \mathbf{R}$. 5

(*d*) (*i*) Let I = $[a, b]$ be a closed and bounded interval and let $f : I \to \mathbf{R}$ be continuous on I. Prove that 'f' has an absolute maximum on I.

(*ii*) Let I = $[a, b]$ be a closed and bounded interval and let $f : I \to \mathbf{R}$ be continuous I such that $f(x) > 0$ for each $x \in I$. Prove that there exists a real number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in I$. 5

5. Attempt any *two* parts :

(*a*) If $f : A \to \mathbf{R}$ is uniformly continuous on a subset A of **R**, show that 'f' is continuous on A. Show by means of an example that a continuous function on A may not be uniformly continuous on A. 3,3

(*b*) Use mean value theorem to prove : 3,3

(*i*) $|\sin x - \sin y| \leq |x - y|$ for all x, y in **R**.

(*ii*) $\dfrac{x-1}{x} < \ln x < x - 1$ for $x > 1$.

(*c*) Suppose that $f : [0, 2] \to \mathbf{R}$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$ and that : 6

$$f(0) = 0, f(1) = 1, f(2) = 1.$$

(*i*) Show that there exists $C_1 \in (0, 1)$ such that $f'(C_1) = 1$.

(*ii*) Show that there exists $C_2 \in (1, 2)$ such that $f'(C_2) = 0$.

(*iii*) Show that there exists $C_3 \in (0, 2)$ such that $f'(C) = 1/3$.

6. Attempt any *two* parts :

(*a*) Obtain Maclaurin's series expansion of :

(*i*) $f(x) = \sin x, x \in \mathbf{R}$

(*ii*) $f(x) = (1 + x)^m, x \in \mathbf{R}$ and $m \in \mathbf{N}$. 5

(*b*) Use Taylor's theorem to approximate $\sqrt[3]{1 + x}, x > -1$ by means of a polynomial of degree 2. Further use this polynomial to obtain approximation for $\sqrt[3]{1.3}$. 5

(c) (i) Approximate the number e with error less than 10^{-5}. 2½

(ii) Define and explain geometrically a convex function on an interval $I \subseteq \mathbf{R}$. Also state the result between a convex function f and its second derivative f'', assuming that f'' exists on I. 2½