This question paper contains 4+1 printed pages]

Your Roll No.....................................

6102

B.Sc. (Hons.)/Sem. I B

MATHEMATICS—Paper 1.2

(Analysis—I)

(Admissions of 2011 and onwards)

Time : 3 *Hours* *Maximum Marks* : 75

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *two* parts from each question.

Use of basic Calculator is allowed.

1. (*a*) Let a, b, c be any elements of **R**. Show that :

(*i*) If $a > b$ and $b > c$, then $a > c$.

(*ii*) If $a > b$, then $a + c > b + c$.

(*iii*) If $a > b$ and $c > 0$, then $ca > cb$. 5

P.T.O.

(*b*) Find all $x \in \mathbf{R}$ that satisfy the inequality :

$$4 < |x + 2| + |x - 1| < 5.$$ 5

(*c*) If $y > 0$, show that there exists $n \in \mathbf{N}$ such that $1/2^n < y$. Justify each step by referring to an appropriate property or theorem. 5

2. (*a*) State and prove the Density Theorem. 5

(*b*) State the Completeness Property of **R**. Show that if A and B are bounded subsets of **R**, then :

$$\sup(A \cup B) = \sup\{\sup A, \sup B\}.$$ 5

(*c*) Show that intersection of any arbitrary collection of closed sets in **R** is closed. Show, by an example, that union of infinitely many closed sets in **R** used not be closed. 5

3. (*a*) Prove that a sequence in **R** can have at most one limit. 5

(*b*) Suppose every subsequence of $X = (x_n)$ has a subsequence that converges to 0. Show that $\lim X = 0$. 5

(*c*) Let $x_1 > 1$ and $x_{n+1} = 2 - \frac{1}{x_n}$ for $n \in \mathbf{N}$. Show that (x_n) is bounded and monotone. Find its limit. 5

4. (*a*) Show that the sequence $(a_n) = ((-1)^n)$ does not converge. 5

(*b*) Let (s_n) be a sequence in $\mathbf{R}$. Prove that $\lim (s_n) = 0$ if and only if $\lim (|s_n|) = 0$. 5

(*c*) Let (s_n) be a sequence that converges. Show that if $s_n \geq a$ for all but finitely many n, then $\lim (s_n) \geq a$. 5

5. (*a*) (*i*) Let $X = (x_n)$ be a bounded increasing sequence. Show that X is convergent and :

$$\lim (x_n) = \sup\{x_n : n \in \mathbf{N}\}$$ 5

(*ii*) Show that $\lim \left(\sqrt{n} + 7\right) = +\infty$. 2½ (*b*) (*i*) Let $X = (x_n)$ be a sequence of real numbers that converges to x and suppose that $x_n \geq 0$. Show that the sequence $\left(\sqrt{x_n}\right)$ of positive square roots converges and $\lim \left(\sqrt{x_n}\right) = \sqrt{x}$. 5

(*ii*) Let (s_n) and (t_n) be the following sequences that repeat in cycles of four :

$(s_n) = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, \ldots\ldots\ldots)$

$(t_n) = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, \ldots\ldots\ldots)$

Find $\liminf (s_n + t_n)$ and $\limsup (s_n + t_n)$. 2½

(*c*) (*i*) Define a Cauchy sequence. Show that the sequence :

$$\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{n}\right)$$

is divergent. 5

(*ii*) Using squeeze theorem or otherwise, determine the limit of the sequence $\left(n^{1/n^2}\right)$. 2½

6\. (a) State and prove the comparison test for series. 5

(b) Test the convergence of :

(i) $\sum \frac{1}{2^n + n}$

(ii) $\sum \frac{(-1)^n n!}{2^n}$. 5

(c) Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges. Also, give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges. Justify your answers. 5

7\. (a) State and prove the Alternating Series Theorem. 5

(b) Test the convergence of :

(i) $\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log \log n)}$

(ii) $\sum \frac{(-1)^n}{n}$. 5

(c) Show that if $\sum a_n$ converges, then $\lim a_n = 0$. Show, by an example, that the converse is not true. 5