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Your Roll No. ....

6103

B.Sc. (Hons.)/I Sem. **B**

MATHEMATICS—Paper 1.3

(Algebra-I)

(Admission of 2011 and onwards)

Time : 3 *Hours* *Maximum Marks* : 75

(Write your Roll No. on the top immediately on receipt of this question paper.)

All six questions are compulsory.

Do any *two* parts from each question.

1. (*a*) Let $p(t)$ be a polynomial over **C** of positive degree n. Prove that $p(t)$ assumes every complex value at least once and assumes all but finitely many complex values n times. 6

(*b*) Let $p(t) = t^{11} + t^8 - 3t^5 + t^4 + t^3 - 2t^2 + t - 2$

(*i*) Show using Descartes' rule of signs that $p(t)$ has at least four non-real zeros.

(*ii*) Estimate an upper bound for the real zeros of $p(t)$. State the result used. 3,3

(*c*) Find the roots of

$$t^4 - t^3 - 7t^2 + 23t - 20 = 0,$$

given that the product of two of the roots is –5. 6

2. (*a*) (*i*) Prove that :

$$\sin 5t = 16 \sin^5 t - 20 \sin^3 t + 5 \sin t.$$

(*ii*) Find the fourth roots of the complex number $z = 1 + i$ and represent them in the complex plane. 3,3.5

(*b*) (*i*) Compute : $z^n + \frac{1}{z^n}$

if $z + \frac{1}{z} = \sqrt{3}$ and $|z| = 1$.

(c) (i) Let

$$A = \{1, 2, 3, 4, 5\} \text{ and}$$

$$P = \{\{1, 2\}, \{3, 4, 5\}\}$$

be a partition of A. List the pairs in the equivalence relation associated with the partition P.

(ii) Define $a \pmod{n}$, where $n > 1$ is a natural number and a is any integer. Find $-381 \pmod{9}$. 3,2

4. (a) Let $S = \{1, 2, 3, 4, 5\}$ and let $f, g : S \rightarrow S$ be the functions defined by :

$$f = \{(1, 2), (2, 1), (3, 4), (4, 5), (5, 3)\}$$

$$g = \{(1, 2), (2, 2), (3, 4), (4, 3), (5, 1)\}$$

(i) Find fog and gof. Are these functions equal ?

(ii) Find the inverse of f if it exists. If it doesn't explain why not ? 3,2

(b) (i) Define countable and uncountable set.

(ii) Prove that cardinality of **N** is equal to the cardinality of $\mathbf{N} \cup \{0\}$, where **N** is the set of all natural numbers. 1,4 (*ii*) Let $M_1(2 - i)$, $M_2(-1 + 2i)$, $M_3(-2 - i)$, $M_4(1 + 2i)$ be four points in the complex plane. Show that the line M_1M_2 is orthogonal to the line M_3M_4. 3,3.5

(*c*) Let z_1, z_2, z_3 be the coordinates of the vertices A_1, A_2, A_3 of a positively oriented equilateral triangle. Prove that :

$$z_1 + \in z_2 + \in^2 z_3 = 0$$

where

$$\in = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}.$$ 6.5

3. (*a*) Let '~' denote an equivalence relation on a set A and $a, b \in A$. Prove that the equivalence classes $\bar{a}$ and $\bar{b}$ are either the same or disjoint. 5

(*b*) For $a, b \in Z|\{0\}$, define $a \sim b$ if and only if $ab > 0$.

(*i*) Prove that '~' defines an equivalence relation.

(*ii*) Find the equivalence classes $\bar{5}$ and $\overline{-5}$.

(*iii*) What is the quotient set determined by this equivalence relation ? 2,2,1

(c) An XYZ club plays with blue chips worth \$ 5.00 and red chips worth \$ 8.00. Use the principle of Mathematical induction to find the largest bet that cannot be made. 5

5. (a) Determine the Existence and Uniqueness of the solutions to the system :

$$\begin{aligned} x_1 + 2x_2 \qquad - 3x_4 + x_5 &= 2 \\ x_1 + 2x_2 + x_3 - 3x_4 + x_5 &= 3 \\ x_1 + 2x_2 \qquad - 3x_4 + 2x_5 &= 4 \\ 3x_1 + 6x_2 + x_3 - 9x_4 + 4x_5 &= 9 \end{aligned}$$

by row reducing the corresponding augmented matrix to reduced echelon form.

List the pivot columns and write the general solution of the system in parametric vector form. 7.5

(b) Determine by inspection (give suitable reasons) or otherwise, if the given set of vectors is linearly dependent or linearly independent :

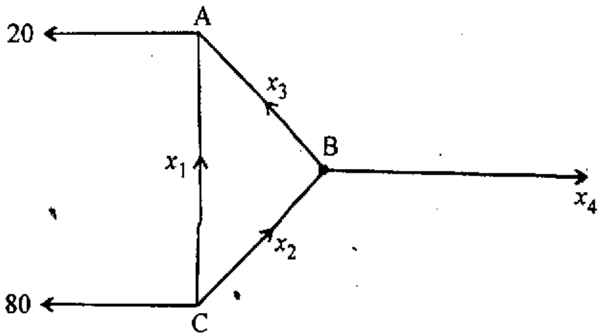
$$S_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} \right\}$$

$$S_2 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \right\}.$$

$$S_3 = \left\{ \begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} \right\}$$

Also, give geometrical description of the span of two vectors in S_3. 7.5

(*c*) (*i*) Find the general flow pattern of the network shown in the figure. Assuming that the flows are all non-negative, what is the largest possible value for x_3 ? 3.5



(*ii*) Find formulae for X, Y, Z in terms of A, B and C when :

$$\begin{bmatrix} X & 0 & 0 \\ Y & 0 & I \end{bmatrix} \begin{bmatrix} A & Z \\ 0 & 0 \\ B & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

where X, Y, Z and A, B, C are matrices of suitable sizes. Justify your calculations. 4

6. (*a*) (*i*) Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be a map defined by

$$T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$$

Show that T is a linear transformation and find its standard matrix.

(*ii*) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation such that :

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \text{ and } T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}.$$

Find the image of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ of $\mathbf{R}^2$. 3.5,4 (*b*) (*i*) Define subspace of $\mathbf{R}^n$. If $v_1, v_2 \in \mathbf{R}^n$, prove that span$\{v_1, v_2\}$ is a subspace of $\mathbf{R}^n$.

(*ii*) Define Null space and column space of a matrix. If

$$A = \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & 5 & 1 \end{bmatrix}$$

find a non-zero vector in Null A and a non-zero vector in Col A. 3,4.5

(*c*) Let

$$v_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, x = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

and H = span$\{v_1, v_2\}$.

(*i*) Show that $\{v_1, v_2\}$ is a basis of H. What is the dimension of subspace H ?

(*ii*) Determine if x is in H, and if it is, find the coordinate vector of x relative to basis $\{v_1, v_2\}$. 3.5,4