In this post, I will use the stationary (time-independent) first order perturbation theory, to find out the relativistic correction to the Energy of the nth state of a Harmonic Oscillator.

The relativistic Kinetic Energy is given as:

\[ T = \sqrt{p^2 c^2 + m^2 c^4} - mc^2 \]

The first term in the above equation is the total energy of a relativistic particle and the second term is the rest mass energy of a particle. So we get the Kinetic energy by subtracting those.

\[ \Rightarrow T = mc^2 \sqrt{1 + \frac{p^2 c^2}{m^2 c^4}} - mc^2 \]

Now, using the binomial expansion on the Kinetic energy we can write it as:

\[ T = mc^2 \left( 1 + \frac{p^2 c^2}{2 m^2 c^4} - \frac{1}{8} \left( \frac{p^2 c^2}{m^2 c^4} \right)^2 + \ldots \right) - mc^2 \]

Note: Binomial Expansion:

\[ (1 + x)^n = 1 + nx + n(n - 1) \frac{x^2}{2} + \ldots \]

\[ \Rightarrow T = (mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3 c^4} + \ldots) - mc^2 \]

If we ignore the rest of the higher order terms and take only the first three terms of the Binomial expansion, then

\[ \Rightarrow T = \frac{p^2}{2m} - \frac{p^4}{8m^3 c^4} \]

We can now see that the Kinetic Energy is actually modified and not just \( \frac{p^2}{2m} \) as in the classical case. Since the second term would be very small due to \( c^2 \) in the denominator, we can take it as a perturbation, and use the time-independent perturbation theory to find out the correction to the energy levels.

Let perturbation, \( H' = -\frac{p^4}{8m^3 c^4} \)

Then the first order energy correction to the nth level is given as:

\[ E_n^1 = \langle \psi_n | H' | \psi_n \rangle \]

\[ \Rightarrow E_n^1 = -\frac{1}{8m^3 c^4} \langle \psi_n | p^4 | \psi_n \rangle \]

From Schrodinger’s Equation:
Using the above relation,

\[ \frac{p^2}{2m} \psi_n \rangle + V \psi_n \rangle = E_n \psi_n \rangle \]

\[ \implies p^2 \psi_n \rangle = 2m(E_n - V) \psi_n \rangle \]

Using the above relation,

\[ E_n^1 = -\frac{1}{8m^3c^2} \langle \psi_n \mid (2m(E_n - V))^2 \rangle \psi_n \rangle \]

\[ E_n^1 = -\frac{1}{2mc^2} \langle \psi_n \mid (E_n^2 + V^2 - 2E_n V) \rangle \psi_n \rangle \]

\[ E_n^1 = -\frac{1}{2mc^2} (E_n^2 + \langle V^2 \rangle - 2E_n \langle V \rangle) \]

From Virial Theorem for Harmonic Oscillator, we know that the expectation value of $V$:

\[ \langle V \rangle = \frac{E_n}{2} \]

\[ \implies E_n^1 = -\frac{1}{2mc^2} (E_n^2 + \langle V^2 \rangle - E_n^2) \]

\[ E_n^1 = -\frac{1}{2mc^2} \langle V^2 \rangle \]

So it all boils down to finding the expectation value of $V^2$.

To do that we would need to use the following relations, for a Harmonic Oscillator:

\[ V = \frac{1}{2}m\omega^2x^2 \]

and

\[ x = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \]

where $\hat{a}$ and $\hat{a}^\dagger$ are annihilation(lowering) and creation(raising) operators respectively.

Substituting the above value of $x$ in the expression for $V$,

\[ V = \frac{1}{2}m\omega^2 \frac{\hbar}{2m\omega}(a + a^\dagger)^2 \]

\[ \implies V^2 = \frac{1}{4}m^2\omega^4 \left( \frac{\hbar}{2m\omega} \right)^2 (\hat{a} + \hat{a}^\dagger)^4 \]

\[ \implies V^2 = \frac{\hbar^2\omega^2}{16}(\hat{a} + \hat{a}^\dagger)^4 \]

Now you might remember the following relations for the operators $\hat{a}$ and $\hat{a}^\dagger$:

\[ \hat{a} \mid n \rangle = \sqrt{n} \mid n - 1 \rangle \]

\[ \hat{a}^\dagger \mid n \rangle = \sqrt{n + 1} \mid n + 1 \rangle \]

where $\mid n \rangle$ is the $n$-th eigenstate of the Harmonic Oscillator.

Therefore, the expectation value of $V^2$ can be found by evaluating the following expression:

\[ \langle V^2 \rangle = \frac{\hbar^2\omega^2}{16} \langle n \mid (\hat{a} + \hat{a}^\dagger)^4 \mid n \rangle \]

Now we don’t need to expand $(\hat{a} + \hat{a}^\dagger)^4$ fully and calculate for all the terms, as only the terms with equal
number of raising and lowering operators, will be finite (non-zero). Different number of raising and lowering operators will lead to a different ket and bra, and since they are orthogonal, their inner product would be zero. Therefore, expanding the above equation and leaving only the non-zero terms we get:

\[
\langle V^2 \rangle = \frac{\hbar^2 \omega^2}{16} \left\langle n \left| \hat{a} \hat{a}^\dagger \hat{a} \hat{a}^\dagger + \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} + \hat{a}^\dagger \hat{a} \hat{a} \hat{a}^\dagger + \hat{a} \hat{a} \hat{a} \hat{a}^\dagger \right| n \right\rangle
\]

\[
\langle V^2 \rangle = \frac{\hbar^2 \omega^2}{16} \left\langle n \left| (n+2)(n+1) + (n+1)^2 + n(n-1) + n^2 + n(n+1) + n(n+1) \right| n \right\rangle
\]

\[
\langle V^2 \rangle = \frac{\hbar^2 \omega^2}{16} \left\langle n \left| 6n^2 + 6n + 3 \right| n \right\rangle
\]

\[
\langle V^2 \rangle = \frac{\hbar^2 \omega^2}{16} (6n^2 + 6n + 3)
\]

Plugging this value of \(\langle V^2 \rangle\) back in \(E_n^1 = -\frac{1}{2mc^2} \langle V^2 \rangle\), we get the relativistic energy correction:

\[
E_n^1 = -\frac{\hbar \omega^2}{32mc^2} (6n^2 + 6n + 3)
\]

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