In this post, I will use the stationary (time-independent) first order perturbation theory, to find out the relativistic correction to the Energy of the nth state of a Harmonic Oscillator.

The relativistic kinetic energy is given as:

$$T = \sqrt{p^2 c^2 + m^2 c^4} - mc^2$$

The first term in the above equation is the total energy of a relativistic particle and the second term is the rest mass energy of a particle. So we get the kinetic energy by subtracting those.

$$\implies T = mc^2 \sqrt{1 + \frac{p^2 c^2}{m^2 c^4}} - mc^2$$

Now, using the binomial expansion on the kinetic energy we can write it as:

$$T = mc^2 \left(1 + \frac{1}{2} \frac{p^2 c^2}{m^2 c^4} - \frac{1}{8} \left(\frac{p^2 c^2}{m^2 c^4}\right)^2 + \ldots\right) - mc^2$$

Note: Binomial Expansion:

$$(1 + x)^n = 1 + nx + n(n - 1)\frac{x^2}{2} + \ldots$$

$$\implies T = \left(mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3 c^4} + \ldots\right) - mc^2$$

If we ignore the rest of the higher order terms and take only the first three terms of the binomial expansion, then

$$\implies T = \frac{p^2}{2m} - \frac{p^4}{8m^3 c^2}$$

We can now see that the kinetic energy is actually modified and not just $\frac{p^2}{2m}$ as in the classical case. Since the second term would be very small due to $c^2$ in the denominator, we can take it as a perturbation, and use the time-independent perturbation theory to find out the correction to the energy levels.

Let perturbation, $H' = -\frac{p^4}{8m^3 c^2}$

Then the first order energy correction to the nth level is given as:

$$E_n^1 = \langle \psi_n | H' | \psi_n \rangle$$

$$\implies E_n^1 = -\frac{1}{8m^3 c^2} \langle \psi_n | p^4 | \psi_n \rangle$$

From Schrodinger’s Equation:

$$\frac{p^2}{2m} | \psi_n \rangle + V | \psi_n \rangle = E_n | \psi_n \rangle$$
\[ p^2 |\psi_n\rangle = 2m(E_n - V) |\psi_n\rangle \]

Using the above relation,

\[ E_n^1 = -\frac{1}{8mc^2} \langle \psi_n | (2m(E_n - V))^2 |\psi_n\rangle \]

\[ E_n^1 = -\frac{1}{2mc^2} \langle \psi_n | (E_n^2 + V^2 - 2E_n V) |\psi_n\rangle \]

\[ E_n^1 = -\frac{1}{2mc^2} (E_n^2 + \langle V^2 \rangle - 2E_n \langle V \rangle) \]

From Virial Theorem for Harmonic Oscillator, we know that the expectation value of \( V \):

\[ \langle V \rangle = \frac{E_n}{2} \]

\[ E_n^1 = -\frac{1}{2mc^2} (E_n^2 + \langle V^2 \rangle - E_n^2) \]

\[ E_n^1 = -\frac{1}{2mc^2} \langle V^2 \rangle \]

So it all boils down to finding the expectation value of \( V^2 \).

To do that we would need to use the following relations, for a Harmonic Oscillator:

\[ V = \frac{1}{2}m\omega^2 x^2 \]

and \( x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) \)

where \( a \) and \( a^\dagger \) are annihilation(lowering) and creation(raising) operators respectively.

Substituting the above value of \( x \) in the expression for \( V \),

\[ V = \frac{1}{2}m\omega^2 \frac{\hbar}{2m\omega} (a + a^\dagger)^2 \]

\[ V^2 = \frac{1}{4} m^2 \omega^4 \left( \frac{\hbar}{2m\omega} \right)^2 (a + a^\dagger)^4 \]

\[ V^2 = \frac{\hbar^2 \omega^2}{16} (a + a^\dagger)^4 \]

Now you might remember the following relations for the operators \( a \) and \( a^\dagger \),

\[ a |n\rangle = \sqrt{n} |n - 1\rangle \]

\[ a^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle \]

where \( |n\rangle \) is the \( n \)-th eigenstate of the Harmonic Oscillator.

Therefore, the expectation value of \( V^2 \) can be found by evaluating the following expression:

\[ \langle V^2 \rangle = \frac{\hbar^2 \omega^2}{16} \langle n | (a + a^\dagger)^4 |n\rangle \]

Now we don’t need to expand \((a + a^\dagger)^4\) fully and calculate for all the terms, as only the terms with equal number of raising and lowering operators, will be finite(non-zero). Different number of raising and lowering operators will lead to a different ket and bra, and since they are orthogonal, their inner product would be zero.
Therefore, expanding the above equation and leaving only the non-zero terms we get:

\[
\langle V^2 \rangle = \frac{\hbar^2 \omega^2}{16} \left\langle n | \hat{a} \hat{a} \hat{a} \hat{a} | n \right\rangle = \frac{\hbar^2 \omega^2}{16} \left\langle n | \hat{a} \hat{a} \hat{a} \hat{a} \hat{a} + \hat{a} \hat{a} \hat{a} \hat{a} + \hat{a} \hat{a} \hat{a} \hat{a} | n \right\rangle
\]

Thus:

\[
\langle V^2 \rangle = \frac{\hbar^2 \omega^2}{16} \left\langle n | (n+2)(n+1) + (n+1)^2 + n(n-1) + n^2 + n(n+1) + n(n+1) | n \right\rangle
\]

\[
\langle V^2 \rangle = \frac{\hbar^2 \omega^2}{16} \left\langle n | 6n^2 + 6n + 3 | n \right\rangle
\]

Plugging this value of \( \langle V^2 \rangle \) back in \( E_n^1 = -\frac{1}{2mc^2} \langle V^2 \rangle \), we get the relativistic energy correction:

\[
E_n^1 = -\frac{\hbar \omega^2}{32mc^2} (6n^2 + 6n + 3)
\]

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